Locally conformally Kähler manifolds

lecture 10: pseudoconvex shells

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LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Conical Kähler manifolds (reminder)

DEFINITION: Let (X, g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m, t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X, g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that $(C(X), gt^2 + dt^2)$ is equipped with a complex structure, in such a way that the conical metric $gt^2 + dt^2$ is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

Kähler potentials and plurisubharmonic functions (reminder)

DEFINITION: A real-valued smooth function on a complex manifold is called **plurisubharmonic (psh)** if the (1,1)-form $dd^c f$ is positive, and **strictly plurisubharmonic** if $dd^c f$ is an Hermitian form.

REMARK: Since $dd^c f$ is always closed, **it is also Kähler when it is strictly positive.**

DEFINITION: Let (M, I, ω) be a Kähler manifold. **Kähler potential** is a function f such that $dd^c f = \omega$.

THEOREM: Let S be a Sasakian manifold, $C(S) = S \times \mathbb{R}^{>0}$ its cone, t the coordinate along the second variable, and $r = t\frac{d}{dt}$. **Then t^2 is a Kähler potential on $C(S)$.**

LCK manifolds with potential (reminder)

DEFINITION: Let M be an LCK manifold, and $(\tilde M,\tilde\omega)$ its Kähler covering. It is called **LCK manifold with potential** if $\tilde M$ admits an automorphic Kähler potential $\varphi:\ \tilde M \longrightarrow \mathbb{R}^{>0}$, $dd^c\varphi = \tilde\omega$, which is **proper** (preimage of a compact is again compact).

THEOREM: The property of being LCK with potential is stable under small deformations.

THEOREM: Let M be an LCK manifold, $\Gamma\subset\mathbb{R}^{>0}$ the monodromy group, and $(\tilde M,\tilde\omega)$ its Kähler covering, with $\tilde M/\Gamma = M$. Assume that $\tilde\omega$ admits a Γ-automorphic Kähler potential φ. **The map φ is proper if and only if $\Gamma = \mathbb{Z}$.**

THEOREM: Let M be an LCK manifold with potential, and $\tilde M$ its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde M_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde M\subset \tilde M_c$. Moreover, the monodromy action on $\tilde M$ is extended to a holomorphic automorphism of $\tilde M_c$.

THEOREM: Let M be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$. **Then M admits a holomorphic embedding to a linear Hopf manifold.**

CR-manifolds

Definition: Let M be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I: B \longrightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_C M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then (B, I) is called **a CR-structure on M**.

Example: A complex manifold is CR, with $B = TM$. Indeed, $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ **is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M \subset X$ a hypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $\operatorname{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, **$M$ is a CR-manifold.**

Definition: **A Frobenius form of a CR-manifold** is the tensor $B \otimes B \longrightarrow TM/B$ mapping X, Y to the $\Pi_{TM/B}([X, Y])$. It is an obstruction to integrability of the foliation given by B.

Contact CR-manifolds.

Complex algebraic geometry is a rich source of contact structures.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. This means that it is Hermitian.

DEFINITION: This Hermitian form is called **Levi form** of a CR-manifold.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Levi form is positive definite.

Example: Let h be a function on a complex manifold such that $\partial\bar\partial h = \omega$ is a positive definite Hermitian form, and $X = h^{-1}(c)$ its level set. Then the Frobenius form of X is equal to $\omega|_X$ (see the next slide). In particular, X **is a strictly pseudoconvex CR-manifold**.

CR-manifolds and plurisubharmonic functions.

PROPOSITION: Let M be a complex manifold, $\varphi \in C^\infty M$ a smooth function, and s a regular value of φ. Consider $S := \varphi^{-1}(s)$ as a CR-manifold, with $B = TS \cap I(TS)$ and let Φ be its Levi form, taking values in

$$TS/B = \ker d\varphi / \ker d\varphi \cap I(\ker d\varphi)$$

Then $d^c\varphi : TS/B \longrightarrow C^\infty S$ trivializes TS/B. Consider tangent vectors $u, v \in B_x S$. **Then $-d^c\varphi(\Phi(u,v)) = dd^c\varphi(x,y)$.**

Proof: Extend u, v to vector fields $u, v \in B = \ker d\varphi \cap I(\ker d\varphi)$. Then $-d^c\varphi(\Phi(u,v)) = -d^c\varphi([u,v]) = dd^c\varphi(u,v)$. ■

COROLLARY: Let M be a complex manifold, $\varphi \in C^\infty M$ a strictly plurisubharmonic function, and s a regular value of φ. **Then $S := \varphi^{-1}(s)$ is strictly pseudoconvex.**

Proof: By the above proposition, the Levi form of S is expressed as $dd^c\varphi(u,v)$, hence it is positive definite. ■

Algebraic cones.

DEFINITION: An algebraic cone is an affine variety $\mathcal{C}$ admitting a $\mathbb{C}^*$-action ρ with a unique fixed point x_0, called **the origin**, and satisfying the following:

(i) $\mathcal{C}$ is smooth outside of x_0,

(ii) ρ acts on the Zariski tangent space $T_{x_0}\mathcal{C}$ with all eigenvalues $|\alpha_i| < 1$.

An open algebraic cone is a closed algebraic cone without the origin.

THEOREM: Let $M = \tilde{M}/A$ be LCK manifold with potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-covering. **Then $\tilde{M}$ is an open algebraic cone.**

Proof. Step 1: Let $\tilde{M}_c$ be a Stein completion of $\tilde{M}$ equipped with an A-equivariant embedding to $\mathbb{C}^n$, where A acts as a linear operator with all eigenvalues $|\alpha_i| < 1$. Denote the ideal of $\tilde{M}_c$ in the local ring $\mathcal{O}_{\mathbb{C}^n,0}$ as I.

Algebraic cones and LCK manifolds with potential

THEOREM: Let $M = \tilde M/A$ be LCK manifold with potential, and $\tilde M$ its Kähler $\mathbb Z$-covering. **Then $\tilde M$ is an open algebraic cone.**

Proof. Step 1: [...] Denote the ideal of $\tilde M_c$ in the local ring $\mathcal O_{\mathbb C^n,0}$ as I.

Step 2: Call an element $f \in \mathcal O_{\mathbb C^n,0}$ **A-finite** if $\langle f, A^* f, A^{2^*} f, ...\rangle$ is finitely-dimensional. A polynomial function is clearly A-finite. The converse is also true, because a Taylor decomposition of an A-finite function f can have only finitely many components, otherwise the eigenspace decomposition of f is infinite. Therefore, **the ideal $I^A := I \cap \mathcal O^A_{\mathbb C^n,0}$ is finitely generated,** where $\mathcal O^A_{\mathbb C^n,0}$ is a ring of A-finite functions (any ideal in the ring of polynomials is finitely generated, by Hilbert basis theorem).

Step 3: As we have shown in Lecture 9, the ring $\mathcal O^A_{\tilde M_c,0}$ is dense in $\mathcal O_{\tilde M_c,0}$ in $\mathfrak m$-adic topology. in other words, it has the same associated graded ring with respect to the $\mathfrak m^n$-filtration as $\mathcal O_{\tilde M_c,0}$. Then the Nakayama's lemma implies that $I = I^A \otimes_{\mathcal O^A_{\mathbb C^n,0}} \mathcal O_{\mathbb C^n,0}$.

Step 4: Let $f_1, ..., f_n \subset \mathcal O_{\mathbb C^n}$ be the polynomial generators of $I \subset \mathcal O_{\mathbb C^n,0}$. Then $\tilde M_c$ is an affine variety defined by the ideal $\langle f_1, ..., f_n\rangle$. ∎

Pseudoconvex shells

DEFINITION: Let $\tilde{M}$ be an open algebraic cone, $\tilde{M}_c$ the corresponding closed cone, and $\vec{r} \in T\mathcal{C}$ a holomorphic vector field such that for all $t > 0$ the diffeomorphism $e^{t\vec{r}}$ is a holomorphic contraction of $\tilde{M}_c$ to origin. A strictly pseudoconvex hypersurface $S \subset \tilde{M}$ is called **a pseudoconvex shell** if S intersects each orbit of $e^{t\vec{r}}$, $t \in \mathbb{R}$ exactly once.

Theorem 1: Let $\tilde{M}$ be an algebraic cone, $e^{t\vec{r}}$ a contraction, and $S \subset \tilde{M}$ a pseudoconvex shell. Then for each $\lambda \in \mathbb{R}$ there exists a unique function φ_λ such that $\operatorname{Lie}_{\vec{r}}\varphi = \lambda\varphi$ and $\varphi_\lambda|_S = 1$. Moreover, **such φ_λ is strictly plurisubharmonic when $\lambda \gg 0$.**

Theorem 2: Any LCK manifold with potential admits a metric of this type.

Theorem 1 (proven later in this lecture) implies the following corollary.

COROLLARY: (Gauduchon-Ornea)
All linear Hopf manifolds are LCK with potential.

Proof: Let $M = (\mathbb{C}^n \backslash 0)/\langle A\rangle$, $\vec{r} = \log A$, and $S \subset \mathbb{C}^n$ be a unit sphere. Then S is a pseudoconvex shell, and for λ sufficiently big a plurisubharmonic function φ_λ gives an LCK-potential. ■

Pseudoconvex shells and plurisubharmonic functions

Theorem 1: Let $\tilde{M}$ be an algebraic cone, $e^{t\vec{r}}$ a contraction, and $S \subset \tilde{M}$ a pseudoconvex shell. Then for each $\lambda \in \mathbb{R}$ there exists a unique function φ_λ such that $\operatorname{Lie}_{\vec{r}}\varphi_\lambda = \lambda\varphi_\lambda$ and $\varphi_\lambda|_S = 1$. Moreover, **such φ_λ is strictly plurisubharmonic when $\lambda \gg 0$.**

Proof. Step 1: For each λ, **φ_λ is uniquely determined on each orbit of $e^{t\vec{r}}$,** $t \in \mathbb{R}$, because φ_λ restricted to this orbit is $e^{\lambda t}$.

Step 2: Let $B := e^{\mathbb{R}\vec{r}} \cdot (TS \cap I(TS)) \subset T\tilde{M}$ be a sub-bundle obtained from $TS \cap I(TS)$ by translations along $e^{t\vec{r}}$. **Then $dd^c\varphi|_B$ is the Levi form of S,** hence it is positive definite.

Step 3: Replacing φ by φ^{2a} amounts to replacing λ by $2a\lambda$. Then

$$dd^c\varphi^{2a} = \varphi^{2a-2}(2a \cdot \varphi dd^c\varphi + 2a(2a-1)d\varphi \wedge d^c\varphi).$$

To prove Theorem 1 it would suffice to show that $dd^c\varphi^{2a}|_S > 0$ for a sufficiently big. However, S is compact, hence it is implied by the following lemma applied to $V = TM$, $W = B$, $h_1 = \varphi dd^c\varphi$, $h_2 = d\varphi \wedge d^c\varphi$.

Positivity of Hermitian forms

LEMMA: Let h_1, h_2 be pseudo-Hermitian forms on a complex vector space V, and $W \subset V$ a subspace of codimension 1. Assume that $h_1|_W$ is strictly positive, $h_2|_W = 0$, and $h_2\Big|_{V/W}$ is also strictly positive. **Then there exists a number $T_0 \in \mathbb{R}$ which depends continuously on h_1, h_2 such that $h_T := h_1 + Th_2$ is positive definite for all $T > T_0$.**

Proof: We think of h_1, h_2 as of real valued bilinear symmetric forms. Let $y \in V$ be a vector which satisfies $h_2(y,y) = 1$. Then any $x \in V$ can be written as $x = ay + z$, $z \in W$. This gives

$$h_T(x,x) = Ta^2 + a^2 h_1(y,y) + h_1(z,z) + 2ah_1(z,y) \quad (*)$$

Consider (*) as a polynomial on a. Then (*) is positive definite for all a if and only if

$$(h_1(z,y))^2 - (T + h_1(y,y)) \cdot h_1(z,z) < 0. \quad (**)$$

Let $y' \in W$ be a vector which satisfies $h_1(z,y') = h_1(z,y)$ for all $z \in W$, and $T > h_1(y',y') - h_1(y,y)$. Then (**) becomes

$$(h_1(z,y'))^2 - h_1(y',y')h_1(z,z) < 0$$

(Cauchy-Schwarz inequality). ∎

Logarithm

DEFINITION: A **Banach ring** is a Banach space equipped with a commutative, continuous product. A Banach ring is **finitely generated** if it is a closure of a finitely-generated ring.

EXAMPLE: A ring of bounded holomorphic function on a complex variety is a Banach ring.

Proposition 1: Let R be a finitely generated, finitely presented Banach ring, and $R_1 \subset R$ a finite-dimensional subspace containing unit, which generates R multiplicatively. We write $R = \mathbb{C}[V]/I$, where I is an ideal and $V = R_1$. Let N a number such that $I \cap V^N$ generates I. Consider an automorphism A of R such that on $R_N := R_1^N$ one has $\|A - \operatorname{Id}\| < 1$, where $\|\cdot\|$ is the operator norm. For each $x \in R_N$, define **the logarithm**: $\log(A)(x) := \sum_{i=1}^\infty \frac{(1-A)^i}{i}(x)$ (the series converges, because $\|A - \operatorname{Id}\| < 1$ on R_N). **Then $\log A$ can be extended to a derivation on R which satisfies $e^{\log A} = \operatorname{Id}$.**

Proof: For each $x, y, xy \in R_N$, one has $\log(A)(xy) = \log(A)(x)y + x\log(A)(y)$ by formal identities with logarithms. Since all relations are generated by elements of $V^N \cap I$, and $\log(A) = 0$ on these by construction, the operator $\log(A)$ can be extended to R using the Leibnitz identity. ∎

Logarithm on LCK manifolds with potential

Lemma 1: Let $\{a_1, ..., a_n\}$ be a finite set of complex numbers which satisfy $0 < |a_i| < 1$. **Then there exists an integer $C > 0$ such that $|a_i^C - 1| < 1$.**

Proof: Write $a_i = b_i u_i$, where $|u_i| = 1$, $b_i \in \mathbb{R}$. For any given ε one can find C such that $\arg(u_i^C) < \varepsilon$ for all i. The statement of the lemma is obtained when $\varepsilon = \frac{\pi}{3}$. ■

THEOREM: Let M be an LCK manifold with potential, $\tilde{M}$ its Kähler its $\mathbb{Z}$-covering, and $M = \tilde{M}/\langle\gamma\rangle$. **Then there exists $C \in \mathbb{Z}^{>0}$ and a holomorphic vector field $\vec{r}$ on $\tilde{M}$ such that $\gamma^C = \vec{r}$.**

Proof: Let $\mathcal{O}^\gamma_{\tilde{M}_c}$ be the ring of γ-finite functions (finitely generated and dense in $H^0(\mathcal{O}_{\tilde{M}_c})$, as shown above), $R_1 = V$ be a set of multiplicative generators of $\mathcal{O}^\gamma_{\tilde{M}_c}$, containing unit, with $R = \mathbb{C}[V]/I$ and N a number such that I is generated by $V^N \cap I$. Define the Banach norm on $\mathcal{O}^\gamma_{\tilde{M}_c}$ by taking $|f| = \sup_{x \in \varphi^{-1}[0,a]} |f(x)|$, where φ is the LCK potential, and let R be its Banach completion. Using Lemma 1, choose $C \in \mathbb{Z}^{>0}$ such that on $R_N := R_1^N$ one has $|\gamma^C - \mathrm{Id}| < 1$, and let $\log \gamma^C$ be the logarithm defined as in Proposition 1. Then $e^{\log \gamma^C} = \gamma^C$, hence we can take $\vec{r} := \log \gamma^C$. ■

$\mathbb{R}$-automorphic LCK metrics

REMARK: Let φ be any LCK-potential on $\tilde M$, satisfying $(\gamma^k)^*\varphi = e^\lambda\varphi$, $\vec r$ a vector field constructed above, and $\rho(t) \longrightarrow e^{-t\lambda}(e^{t\vec r})^*$ the corresponding endomorphism of $C^\infty M$. Since $\rho(k+t)(\varphi) = \rho(t)\varphi$, the orbit of φ is compact. **Averaging $\rho(t)\varphi$ over $\mathbb{R}$, we obtain a $\rho(t)$-invariant Kähler potential φ_0.** Then φ_0 is obtained from a pseudoconvex shell $\varphi_0^{-1}(1)$ and the vector field $\vec r$ as in Theorem 1. **This proves Theorem 2.**