Locally conformally Kähler manifolds

lecture 11: CR-geometry of Sasakian manifolds

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LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: **A manifold is locally conformally Kähler** iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: **Deck transform**, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: **An LCK manifold** is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: **These three definitions are equivalent**.

Conical Kähler manifolds (reminder)

DEFINITION: Let (X, g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m,t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X, g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that $(C(X), gt^2+dt^2)$ is equipped with a complex structure, in such a way that the conical metric gt^2+dt^2 is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

Reeb field (reminder)

DEFINITION: A Sasakian manifold is a contact manifold S with a Riemannian structure, such that the symplectic cone $C(S)$ with its Riemannian metric is Kähler.

DEFINITION: Let S be a Sasakian manifold, ω the Kähler form on $C(S)$, and $r = t\frac{d}{dt}$ the homothety vector field. Then $\operatorname{Lie}_{Ir} t = \langle dt, Ir\rangle = 0$, hence iR is tangent to $S \subset C(S)$. This vector field (denoted by Reeb) is called **the Reeb field** of a Sasakian manifold.

REMARK: The Reeb field is dual to the contact form $\theta = \omega \lrcorner r$.

THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.

DEFINITION: A Sasakian manifold is called **regular** if the Reeb field generates a free action of S^1, **quasiregular** if all orbits of Reeb are closed, and **irregular** otherwise.

Vaisman manifolds (reminder)

EXAMPLE: For any given $\lambda \in \mathbb{R}^{>1}$, **the quotient $C(X)/h_\lambda$ of a conical Kähler manifold is locally conformally Kähler.**

DEFINITION: An LCK manifold (M, g, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection associated with g.

THEOREM: Let M be a Vaisman manifold, $\tilde{M}$ its covering; the pullback of the Lee form θ to $\tilde{M}$ is denoted by the same letter θ. Assume that $d\psi = \theta$ on $\tilde{M}$ (such ψ exists, for example, if $\tilde{M}$ is a universal cover of M). Consider the form $\tilde{\omega} := e^{-\psi}\omega$. **Then $(\tilde{M}, \tilde{\omega})$ is a Kähler manifold, isometric to a cone.**

THEOREM: **Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$,** where X is Sasakian, $\mathbb{Z} = \left\langle (x,t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and φ is a Sasakian automorphism of X. Moreover, the triple (X, φ, q) is unique.

LCK manifolds with potential (reminder)

DEFINITION: Let M be an LCK manifold, and $(\tilde M, \tilde\omega)$ its Kähler covering. It is called **LCK manifold with potential** if $\tilde M$ admits an automorphic Kähler potential $\varphi: \tilde M \longrightarrow \mathbb{R}^{>0}$, $dd^c\varphi = \tilde\omega$, which is **proper** (preimage of a compact is again compact).

THEOREM: The property of being LCK with potential is stable under small deformations.

THEOREM: Let M be an LCK manifold, $\Gamma \subset \mathbb{R}^{>0}$ the monodromy group, and $(\tilde M, \tilde\omega)$ its Kähler covering, with $\tilde M/\Gamma = M$. Assume that $\tilde\omega$ admits a Γ-automorphic Kähler potential φ. **The map φ is proper if and only if $\Gamma = \mathbb{Z}$.**

THEOREM: Let M be an LCK manifold with potential, and $\tilde M$ its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde M_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde M \subset \tilde M_c$. Moreover, the monodromy action on $\tilde M$ is extended to a holomorphic automorphism of $\tilde M_c$.

THEOREM: Let M be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$. **Then M admits a holomorphic embedding to a linear Hopf manifold.**

CR-manifolds (reminder)

Definition: Let M be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I: B \longrightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_C M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then (B, I) is called **a CR-structure on M**.

Example: A complex manifold is CR, with $B = TM$. Indeed, $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ **is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M \subset X$ a hypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $\operatorname{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, **$M$ is a CR-manifold.**

Definition: **A Frobenius form of a CR-manifold** is the tensor $B \otimes B \longrightarrow TM/B$ mapping X, Y to the $\Pi_{TM/B}([X, Y])$. It is an obstruction to integrability of the foliation given by B.

Contact CR-manifolds (reminder).

Definition: Let (M,B,I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. This means that it is Hermitian.

DEFINITION: This Hermitian form is called **Levi form** of a CR-manifold.

Definition: Let (M,B,I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Levi form is positive definite.

PROPOSITION: Let M be a complex manifold, $\varphi \in C^\infty M$ a smooth function, and s a regular value of φ. Consider $S := \varphi^{-1}(s)$ as a CR-manifold, with $B = TS \cap I(TS)$ and let Φ be its Levi form, taking values in TS/B. Then $d^c\varphi: \ TS/B \longrightarrow C^\infty S$ trivializes TS/B. Consider tangent vectors $u, v \in B_x S$. **Then $-d^c\varphi(\Phi(u,v)) = dd^c\varphi(x,y))$.**

COROLLARY: Let M be a complex manifold, $\varphi \in C^\infty M$ a strictly plurisubharmonic function, and s a regular value of φ. **Then $S := \varphi^{-1}(s)$ is strictly pseudoconvex.**

Algebraic cones (reminder).

DEFINITION: An algebraic cone is an affine variety $\mathcal{C}$ admitting a $\mathbb{C}^*$-action ρ with a unique fixed point x_0, called **the origin**, and satisfying the following:

(i) $\mathcal{C}$ is smooth outside of x_0,

(ii) ρ acts on the Zariski tangent space $T_{x_0}\mathcal{C}$ with all eigenvalues $|\alpha_i| < 1$.

An open algebraic cone is a closed algebraic cone without the origin.

THEOREM: Let $M = \tilde{M}/A$ be LCK manifold with potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-covering. **Then $\tilde{M}$ is an open algebraic cone.**

Pseudoconvex shells and logarithm (reminder)

DEFINITION: Let $\tilde{M}$ be an open algebraic cone, $\tilde{M}_c$ the corresponding closed cone, and $\vec{r} \in T\mathcal{C}$ a holomorphic vector field such that for all $t > 0$ the diffeomorphism $e^{t\vec{r}}$ is a holomorphic contraction of $\tilde{M}_c$ to origin. A strictly pseudoconvex hypersurface $S \subset \tilde{M}$ is called **a pseudoconvex shell** if S intersects each orbit of $e^{t\vec{r}}$, $t \in \mathbb{R}$ exactly once.

Theorem 1: Let $\tilde{M}$ be an algebraic cone, $e^{t\vec{r}}$ a contraction, and $S \subset \tilde{M}$ a pseudoconvex shell. Then for each $\lambda \in \mathbb{R}$ there exists a unique function φ_λ such that $\operatorname{Lie}_{\vec{r}}\varphi_\lambda = \lambda\varphi_\lambda$ and $\varphi_\lambda|_S = 1$. Moreover, **such φ_λ is strictly plurisubharmonic when $\lambda \gg 0$.**

COROLLARY: (Gauduchon-Ornea)
All linear Hopf manifolds are LCK with potential.

THEOREM: Let M be an LCK manifold with potential, $\tilde{M}$ its Kähler its $\mathbb{Z}$-covering, and $M = \tilde{M}/\langle\gamma\rangle$. **Then there exists $C \in \mathbb{Z}^{>0}$ and a holomorphic vector field $\vec{r}$ on $\tilde{M}$ such that $\gamma^C = \vec{r}$.**

Proof: Lecture 10.

DEFINITION: Such a vector field is called **a logarithm of monodromy**.

Theorem of Kamishima-Ornea

THEOREM: (Kamishima-Ornea)
Let (M,ω,θ) be an LCK manifold equipped with a holomorphic conformal $\mathbb{C}$-action, which lifts to non-isometric homotheties on its Kähler covering $\tilde M$. **Then (M,ω,θ) is conformally equivalent to a Vaisman manifold.**

Proof. Step 1: Let $\vec r$ be a vector field of this $\mathbb{C}$-action, and $\tilde\omega$ the Kähler form of $\tilde M$. Then $\operatorname{Lie}_{\vec r}\tilde\omega = a\tilde\omega$ and $\operatorname{Lie}_{I\vec r}\tilde\omega = b\tilde\omega$. Replacing $\vec r$ by a linear combination of $\vec r$ and $I(\vec r)$, we obtain a vector field preserving $\tilde\omega$. Replacing $\vec r$ by an appropriate linear combination of $\vec r$ and $I\vec r$, **we can assume that $\operatorname{Lie}_{I\vec r}\tilde\omega = 0$ and $\operatorname{Lie}_{\vec r}\tilde\omega = \tilde\omega$.**

Step 2: $\operatorname{Lie}_{\vec r}\tilde\omega = d(\tilde\omega \lrcorner \vec r) = \tilde\omega$, and $\operatorname{Lie}_{I\vec r}\tilde\omega = d\eta = 0$, where $\eta = I(\omega \lrcorner \vec r) = \omega \lrcorner (I\vec r)$.

Step 3: $\operatorname{Lie}_{\vec r}\eta = \eta = d(\eta \lrcorner \vec r) = d\langle \eta, \vec r\rangle$. This gives $\tilde\omega = dd^c\varphi$, where $\varphi = \langle \eta, \vec r\rangle$.

Step 4: The action by $\vec r$ multiplies the Kähler potential φ by a constant; the action by $I(\vec r)$ preserves φ. Therefore, **$\tilde M$ is locally isometric to a Kähler cone,** and $\omega := \varphi^{-1}\tilde\omega$ is Vaisman (Lecture 3), that is, satisfies $\nabla\theta = 0$. ■

Vaisman manifolds and homothety action

DEFINITION: Let $\tilde M$ be an algebraic cone, $\rho := e^{t\vec r}$ a contraction, and $S \subset \tilde M$ a pseudoconvex shell. Consider the (necessarily unique) function potential φ_λ which satisfies $\operatorname{Lie}_{\vec r}\varphi_\lambda = \lambda\varphi_\lambda$. Assume that it is a Kähler potential (by Theorem 1, it is a Kähler potential for $\lambda \gg 0$). Then φ_λ is called **ρ-automorphic Kähler potential**, and $dd^c\varphi$ **ρ-automorphic Kähler form**.

THEOREM: Let (M,ω) be an LCK-manifold with potential, and $\tilde M$ its algebraic cone, $\tilde M/\langle\gamma\rangle = M$, and φ its Kähler potential. Then **ω is conformally equivalent to a Vaisman metric if and only if there exists a logarithm $\vec r$ of γ such that $\operatorname{Lie}_{I\vec r}\varphi = 0$.**

Proof: If M is Vaisman, $\tilde M = C(S)$, where S is Sasakian, $\vec r := t\frac{d}{dt}$ its logarithm, and $I\vec r$ its Reeb field, acting on $C(S)$ by holomorphic isometries.

Conversely, if $\tilde M$ admits a logarithm $\vec r$ with such properties, then the corresponding holomorphic flow acts on $\tilde M$ by homotheties, and M is Vaisman by Kamishima-Ornea. ■

Stein manifolds (reminder).

DEFINITION: A complex variety M is called **holomorphically convex** if for any infinite discrete subset $S \subset M$, there exists a holomorphic function $f \in \mathcal{O}_M$ which is unbounded on S.

DEFINITION: A complex variety is called **Stein** if it is holomorphically convex, and has no compact complex subvarieties.

REMARK: Equivalently, **a complex variety is Stein if it admits a closed holomorphic embedding into $\mathbb{C}^n$.**

THEOREM: (K. Oka, 1942) **A complex manifold M is Stein** if and only M admits a Kähler metric with a **Kähler potential which is positive and proper** (proper = preimages of compact sets are compact).

THEOREM: (H. Cartan, 1951) **A complex variety M is Stein** if and only if for any coherent sheaf F on M, **its cohomology $H^i(F)$ vanish for all $i > 0$.**

CR-holomorphic functions and vector fields

DEFINITION: Let (S, B, I) be a CR-manifold. A function f on S is called **CR-holomorphic** if for any vector field $v \in B^{0,1}$, we have $\operatorname{Lie}_v f = 0$. A vector field $v \in TM$ is called **CR-holomorphic** if the corresponding diffeomorphism flow preserves B and I.

THEOREM: (Rossi-Andreotti-Siu)
Let S be a compact strictly pseudoconvex CR-manifold, $\dim_{\mathbb{R}} S \geqslant 5$, and $H^0(\mathcal{O}_S)_b$ the ring of bounded CR-holomorphic functions. **Then S is a boundary of a Stein manifold M with isolated singularities,** such that $H^0(\mathcal{O}_S)_b = H^0(\mathcal{O}_M)_b$, where $H^0(\mathcal{O}_M)_b$ denotes the ring of bounded holomorphic functions. Moreover, M is defined uniquely, $M = \operatorname{Spec}(H^0(\mathcal{O}_S)_b)$.

COROLLARY: The Lie group $G := \operatorname{Aut}(S)$ of CR-automorphisms is identified with the group of complex automorphisms of the corresponding Stein space M. **Its Lie algebra (the algebra of holomorphic vector fields) is the Lie algebra of holomorphic vector fields on M.**

Burns-Lee theorem

THEOREM: (Dan Burns, John M. Lee)
Let S be a compact strictly pseudoconvex CR-manifold, and $\operatorname{Aut}_0(S)$ the connected component of its group of automorphisms. **Then** $\operatorname{Aut}_0(S)$ **is compact** unless S is equivalent to the standard sphere $S^{2n-1} \subset \mathbb{C}^n$ with its induced CR-structure. In the latter case $\operatorname{Aut}_0(S) = U(1,n)$.

This theorem would not be used.

CR-manifolds and Sasakian manifolds

DEFINITION: Let (S, B, I) be a CR-manifold. We say that S **admits a Sasakian structure** if it can be realized as a CR-hypersurface $S \subset C(S)$, where $C(S)$ is a conical Kähler manifold.

DEFINITION: Let (S, B, I) be a CR-manifold, with TS/B oriented (for strictly pseudoconvex CR-manifolds, the Levi form defines the orientation on TS/B). A vector field $v \in TS$ is called **positive** if it is transversal to B everywhere, and its projection to TS/B is positive.

EXAMPLE: The Reeb field of a Sasakian manifold is always positive (or negative, depending on the choice of orientation). Indeed, I Reeb is always normal to S, hence Reeb $\notin B = TS \cap I(TS)$.

THEOREM: Let S be a strictly pseudoconvex compact CR-manifold, $\dim_{\mathbb{R}} S \geqslant 5$. **Then S admits a Sasakian structure if and only if S admits a positive holomorphic vector field.** This vector field becomes a Reeb field of this Sasakian manifold.

REMARK: The implication "admits a Sasakian structure" $\Rightarrow$ "admits a positive holomorphic vector field" is clear, because the Reeb vector field is positive and CR-holomorphic.

CR-manifolds and Sasakian manifolds (2)

Assume that a strictly pseudoconvex compact CR-manifold S admits a CR-holomorphic positive vector field R. **We need to construct a Sasakian metric on S such that R is its Reeb field.**

REMARK: The argument here is essentially the same as used to embed an LCK manifold with potential to a Hopf manifold.

Step 1: By Rossi-Andreotti-Siu, $S = \partial M$, where $M = \operatorname{Spec}(H^0(\mathcal{O}_S)_b)$ is a Stein variety with isolated singularities, and R acts on M by holomorphic automorphisms.

Step 2: Since R is positive, IR is transversal to ∂M; replacing R by $-R$, we can always assume that IR points toward interior of M, and $A_\varepsilon := e^{\varepsilon IR}$ **for small ε maps M to a subset $A_\varepsilon(M) \subset M$ with compact closure.**

Step 3: Consider the ring $\mathcal{H} = H^0(\mathcal{O}_M)_b$ of bounded holomprhic functions on M, with sup-metric. Then $\mathcal{H}$ is a Banach ring. Since $A_\varepsilon(M)$ has compact closure, $A_\varepsilon^*\mathcal{H}$ **is a normal family, and A_ε^* is a compact operator.**

CR-manifolds and Sasakian manifolds (3)

Step 4: By maximum principle, for any non-constant $f \in \mathcal{H}$, one has $\sup_{A_\varepsilon(M)}|f| < \sup_M|f|$. **Since any limit point $f_{\lim}$ of a sequence $(A_\varepsilon^i)^* f$ satisfies $\sup A_\varepsilon(M)|f| = \sup_M|f|$, it is constant.** A limit function $f_{\lim}$ exists, because $A_\varepsilon^*\mathcal{H}$ is precompact.

Step 5: This implies that for each $z \in M$, a limit point $z_{\lim}$ of a sequence $\{z, A_\varepsilon z, A_\varepsilon^2 z, ...\}$ is unique and independent of z. Indeed, $f_{\lim}(z) = f(z_{\lim})$, but $f_{\lim} = const$. This implies that **A_ε is a holomorphic contraction contracting M to the origin point $x_0 \subset M$.**

Step 6: Since $R\Big|_{S_\varepsilon} = A_\varepsilon(R)$ is nowhere vanishing for each ε, the vector field $\vec{r} := IR$ is transversal to $S_\varepsilon := A_\varepsilon(S)$ pointing to the origin. Therefore, through each point of S passes a unique solution $\rho(t)$ of an equation $\frac{d\rho(t)}{dt} = \vec{r}$.

Step 7: Let φ_λ be a ρ-automorphic Kähler potential associated with this S and ρ as above. **The Lie algebra $\langle R, IR\rangle$ acts on $(M, dd^c\varphi_\lambda)$ by holomorphic homotheties, hence it is a conical Kähler manifold** (Kamishima-Ornea). Therefore, S is Sasakian. ■

Jordan-Chevalley decomposition

DEFINITION: An algebraic group is a group object in the category of affine schemes. **A pro-algebraic group** is an inverse limit of algebraic groups. Further on, all algebraic groups are considered over $\mathbb{C}$.

DEFINITION: An element of an algebraic group G is called **semisimple** if its image is semisimple for any algebraic representation of G, and **unipotent** if its image is unipotent (that is, exponent of nilpotent) for any algebraic representation of G

THEOREM: (The Jordan-Chevalley decomposition)
Let G be an algebraic group, and $a \in G$. **Then there exists a unique decomposition $A = SU$ of A onto a product of commuting elements S and U, where U is unipotent and S semisimple.**

EXERCISE: Prove this theorem.

REMARK: Since this decomposition is unique, it is functorial. **Therefore, it is also true for all pro-algebraic groups.**

Semisimple LCK manifolds with potential

Recall that a linear operator is called **semisimple** if it is diagonalizable over an algebraic closure of the basic field.

DEFINITION: Let M be an LCK manifold with potential, and $j: M \longrightarrow H$ a holomorphic embedding to a Hopf manifold $H = \mathbb{C}^n \backslash 0/\langle A\rangle$. Then M is called **semisimple** if A is semisimple.

REMARK: Let M be an LCK manifold with potential, $\tilde M$ its Kähler $\mathbb{Z}$-covering, $M = \tilde M/\langle A\rangle$, $j: M \longrightarrow H$ a holomorphic embedding to a Hopf manifold $H = \mathbb{C}^n \backslash 0/\langle A\rangle$, and $\tilde M_c$ its completion. Let R be a $\mathfrak{m}$-adic completion of $\mathcal{O}_{\tilde M_c}$ in the maximal ideal of the origin $x_0 \in \tilde M_c$, $R = \lim_{\leftarrow} \mathcal{O}_{\tilde M_c}/\mathfrak{m}^k$. Let $G := \operatorname{Aut}(R)$; clearly,

$$G = \lim_{\leftarrow} \operatorname{Aut}(\mathcal{O}_{\tilde M_c}/\mathfrak{m}^k),$$

hence it is a pro-algebraic group.

Semisimple LCK manifolds with potential (2)

PROPOSITION: Let $j: M \longrightarrow H$ be an embedding of an LCK manifold $M = \tilde{M}/\langle A\rangle$ to a Hopf manifold $H = V\backslash 0/\langle A\rangle$, such that V is an A-invariant subspace in $\mathcal{O}_{\tilde{M}_c}$. **Then the action of A on V is semisimple if and only if A is semisimple as an element of the proalgebraic group $G = \lim\limits_{\leftarrow} \operatorname{Aut}(\mathcal{O}_{\tilde{M}_c}/\mathfrak{m}^k)$.**

Proof: If A is semisimple as an element of G, its action on V, considered as an A-invariant subspace in $R \supset \mathcal{O}_{\tilde{M}_c}$, is also semisimple.

Conversely, if A is semisimple on V, $\mathcal{O}_{\tilde{M}_c}$ is a subring in R, which is a quotient ring of $\mathbb{C}[[V]]$; the latter is an adic completion of the polynomial ring $\mathbb{C}[V]$, where A is clearly semisimple. ■

Vaisman manifolds are semisimple

THEOREM: Let M be an LCK manifold with potential. Then **M is semisimple $\Leftrightarrow$ it admits a Vaisman structure.**

Proof of Vaisman $\Rightarrow$ semisimple:
Let M be Vaisman, $M = \tilde{M}/\langle A\rangle$, and $\tilde{M}_c$ its completion, equipped with the Kähler potential $\varphi : \tilde{M}_c \longrightarrow \mathbb{R}^{\geqslant 0}$. Consider a compact subset $\tilde{M}_c^a := \varphi^{-1}([0,a])$. Consider an L^2-structure on the ring $H^0(\mathcal{O}_{\tilde{M}_c^a})_b$ of bounded holomorphic functions, $|f|^2 = \int_{\tilde{M}_c^a} |f|^2\tilde{\omega}^n$. Let $\vec{r} := t\frac{d}{dt}$ be the homothety vector field on $\tilde{M} = C(S)$. **Then $I\vec{r}$ acts on $H^0(\mathcal{O}_{\tilde{M}_c^a})_b$ by isometries,** hence its action on each finite-dimensional subspace of $H^0(\mathcal{O}_{\tilde{M}_c^a})_b$ is semisimple.

Semisimple LCK manifolds are Vaisman

Proof of semisimple $\Rightarrow$ Vaisman. Step 1:
Since all subvarieties of Vaisman manifolds are again Vaisman, **it would suffice only to show that all semisimple Hopf manifolds are Vaisman.**

Step 2: Let $H = V\backslash 0/\langle A\rangle$ be a semisimple Hopf manifold, e_i an eigenvalue basis in V, and $A(e_i) = \alpha_i u_i e_i$, with $\alpha_i \in]0,1[$ and $u_i \in U(1)$. Consider a unit sphere $S \subset V = C^n$, and let $\rho(t)(e_i) := \alpha_i^t e_i$. Then there exists a ρ-automorphic Kähler potential φ_λ on $V\backslash 0$. Since $A \circ \rho(-1)$ preserves S and A commutes with $\rho(t)$, the function φ_λ is A-automorphic.

Step 3: By Kamishima-Ornea, **this metric is Vaisman whenever S** (and, therefore, the potential φ_λ, and the corresponding automorphic Kähler form) **is invariant with respect to $e^{tI\vec{r}}$,** where $r = \frac{d\rho(t)}{dt}$. However, $e^{tI\vec{r}}(e_i) = e^{\sqrt{-1}\, t \log \alpha_i} e_i$, and this operator is unitary. ■