Locally conformally Kähler manifolds

lecture 12: Morse-Novikov cohomology

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Local systems (reminder)

DEFINITION: A **local system** is a locally constant sheaf of vector spaces.

THEOREM: A local system with fiber B at $x \in M$ gives a homomorphism $\pi_1(M, x) \longrightarrow \operatorname{Aut}(B)$. **This correspondence gives an equivalence of categories.**

DEFINITION: A bundle (B, ∇) is called **flat** if its curvature vanishes.

DEFINITION: A section b of (B, ∇) is called **parallel** if $\nabla(b) = 0$.

CLAIM: Let (B, ∇) be a flat bundle on M, and $\mathcal{B}$ be the sheaf of parallel sections. **Then $\mathcal{B}$ is a locally constant sheaf.**

THEOREM: This correspondence **gives an equivalence of categories** of flat bundles and local systems.

LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Conical Kähler manifolds (reminder)

DEFINITION: Let (X,g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m,t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X,g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that $(C(X), gt^2 + dt^2)$ is equipped with a complex structure, in such a way that the conical metric $gt^2 + dt^2$ is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

Vaisman manifolds (reminder)

EXAMPLE: For any given $\lambda \in \mathbb{R}^{>1}$, **the quotient $C(X)/h_\lambda$ of a conical Kähler manifold is locally conformally Kähler.**

DEFINITION: An LCK manifold (M, g, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection associated with g.

THEOREM: Let M be a Vaisman manifold, $\tilde M$ its covering; the pullback of the Lee form θ to $\tilde M$ is denoted by the same letter θ. Assume that $d\psi = \theta$ on $\tilde M$ (such ψ exists, for example, if $\tilde M$ is a universal cover of M). Consider the form $\tilde\omega := e^{-\psi}\omega$. **Then $(\tilde M, \tilde\omega)$ is a Kähler manifold, isometric to a cone.**

THEOREM: Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$, where X is Sasakian, $\mathbb{Z} = \left\langle (x,t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and φ is a Sasakian automorphism of X. Moreover, the triple (X, φ, q) is unique.

LCK manifolds with potential (reminder)

DEFINITION: Let M be an LCK manifold, and $(\tilde M, \tilde\omega)$ its Kähler covering. It is called **LCK manifold with potential** if $\tilde M$ admits an automorphic Kähler potential $\varphi: \tilde M \longrightarrow \mathbb{R}^{>0}$, $dd^c\varphi = \tilde\omega$, which is **proper** (preimage of a compact is again compact).

THEOREM: **The property of being LCK with potential is stable under small deformations.**

THEOREM: Let M be an LCK manifold, $\Gamma \subset \mathbb{R}^{>0}$ the monodromy group, and $(\tilde M, \tilde\omega)$ its Kähler covering, with $\tilde M/\Gamma = M$. Assume that $\tilde\omega$ admits a Γ-automorphic Kähler potential φ. **The map φ is proper if and only if $\Gamma = \mathbb{Z}$.**

THEOREM: Let M be an LCK manifold with potential, and $\tilde M$ its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde M_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde M \subset \tilde M_c$. Moreover, the monodromy action on $\tilde M$ is extended to a holomorphic automorphism of $\tilde M_c$.

THEOREM: Let M be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$. **Then M admits a holomorphic embedding to a linear Hopf manifold.**

Semisimple LCK manifolds with potential (reminder)

Recall that a linear operator is called **semisimple** if it is diagonalizable over an algebraic closure of the basic field.

DEFINITION: Let M be an LCK manifold with potential, and $j: M \longrightarrow H$ a holomorphic embedding to a Hopf manifold $H = \mathbb{C}^n \backslash 0/\langle A\rangle$. Then M is called **semisimple** if A is semisimple.

PROPOSITION: Let $j: M \longrightarrow H$ be an embedding of an LCK manifold $M = \tilde{M}/\langle A\rangle$ to a Hopf manifold $H = V\backslash 0/\langle A\rangle$, such that V is an A-invariant subspace in $\mathcal{O}_{\tilde{M}_c}$. **Then the action of A on V is semisimple if and only if A is semisimple as an element of the proalgebraic group $G = \varprojlim \operatorname{Aut}(\mathcal{O}_{\tilde{M}_c}/\mathfrak{m}^k)$.**

THEOREM: Let M be an LCK manifold with potential. Then **M is semisimple $\Leftrightarrow$ it admits a Vaisman metric.**

Morse-Novikov cohomology

DEFINITION: Define **the B-valued de Rham differential** $d_\nabla : \Lambda^i(M) \otimes B \longrightarrow \Lambda^{i+1}(M) \otimes B$ as $d_\nabla(\eta \otimes b) := d\eta \otimes b + (-1)^{\tilde\eta} \eta \wedge \nabla b$. **It is easy to check that $d_\nabla^2 = 0$ if and only if the curvature of ∇ vanishes.**

CLAIM: The cohomology of the complex $(\Lambda^* M \otimes B, d_\nabla)$ are equal to the cohomology of the local system $\mathcal{B} := \ker \nabla$.

Proof: The complex of sheaves

$$\Lambda^0(M) \otimes B \xrightarrow{d_\nabla} \Lambda^1(M) \otimes B \xrightarrow{d_\nabla} \Lambda^2(M) \otimes B$$

is a fine resolvent for the sheaf $\mathbb{B}$ of parallel section of (B, ∇), hence its cohomology is $H^i(M, \mathbb{B})$. ■

REMARK: Let B be a line bundle equipped with a flat connection, φ its trivialization, and θ its connection form, $\nabla(f\varphi) = df \otimes \varphi + f\theta \otimes \psi$. Then $d_\nabla(\eta \otimes \psi) = d\eta \otimes \psi + \theta \wedge \eta \otimes \psi$. This is written as $d_\nabla = d + \theta$.

DEFINITION: Cohomology of the complex $(\Lambda^* M, d - \theta)$ are called **Morse-Novikov cohomology**, or **Lichnerowicz cohomology**; the corresponding complex – **Morse-Novikov complex**. They compute the cohomology of the local system L^{-1}

Automorphic forms

DEFINITION: Let M be a manifold, $\tilde{M}$ its Galois covering. A form η on $\tilde{M}$ is called **automorphic** if for any $\gamma \in \pi_1(M)$ acting on $\tilde{M}$ as usual, the form $\gamma^*\eta$ is proportional to η. The character $\chi_\eta(\gamma) := \frac{\gamma^*\tilde{\eta}}{\tilde{\eta}}$ is called **the character of automorphy**, or **weight** for η.

DEFINITION: Let L be an oriented real line bundle equipped with a flat connection (we call L **weight bundle**), and $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ its monodromy. **Automorphic form of weight λ** is an automorphic form which satisfies $\gamma^*\tilde{\eta} = \lambda^{-1}\chi(\gamma)\eta$ for each $\gamma \in \pi_1(M)$. We denote the space of such forms by $\Lambda^*(M)_\lambda$.

DEFINITION: Let (L, ∇) be a flat, oriented line bundle, and φ a nowhere degenerate section, trivializing L. Then $\nabla(f\varphi) = f\theta \otimes \varphi + df \otimes \varphi$. The form θ is called **connection 1-form of ∇**. For any real number λ, we define **the tensor power L^λ** as a flat bundle with a connection form $\lambda\theta$. Since $\nabla(\varphi^k) = k\theta \otimes \varphi^k$, this definition is compatible with the usual one.

REMARK: Let (L, ∇) be a weight bundle, φ its trivialization, and θ the corresponding connection form. **Then θ is closed.** Indeed, flatness of ∇ means that $0 = d^2_\nabla(v) = d\theta \otimes v$ for any section $v \in L$.

Automorphic forms (2)

PROPOSITION: Let (L,∇) be a weight bundle on a manifold, φ its trivialization, and θ a connection form. Denote by $\tilde{M} \xrightarrow{\pi} M$ the universal covering, and let Φ be a non-zero parallel section of π^*L. Then

a. $\frac{\pi^*\varphi}{\Phi}$ **is an automorphic function of weight λ.**

b. Moreover, for each L^λ-valued differential form η, the differential form $\frac{\pi^*\eta}{\pi^*\varphi} \in \Lambda^*(\tilde{M})$ is automorphic of weight $-\lambda$, giving **equivalence Ξ between the space of sections of $\Lambda^*(M)\otimes L$ and $\Lambda^*(M)_\lambda$.**

c. Under this equivalence, **de Rham differential on $\Lambda^*(M)_\lambda$ corresponds to d_∇.**

Proof: (a) is clear, because monodromy acts on $\pi^*\varphi$ trivially and on Φ with weight λ. (b) is clear by the same reason: any section of $\Lambda^*(M)\otimes L$ produces an automorphic form, and any automorphic form ρ gives a section $\rho\Phi^{-1}$ of π^*L^λ which is fixed by mondromy, hence obtained as a pullback.

To prove (c), take any $\rho\in\Lambda^*(M)\otimes L$, then

$$d(\Xi(\rho)) = d(\pi^*\rho\varphi^{-1}) = \pi^* d_\nabla \rho\varphi^{-1} - \rho\wedge\nabla\varphi\cdot\varphi^{-1} = \Xi(d_\nabla\rho) - \Xi(\rho\wedge\theta)$$

where θ is a connection form in L. ■

REMARK: We obtain that **Morse-Novikov complex is identified with the de Rham complex of automorphic forms on $\tilde{M}$.**

Morse-Novikov Dolbeault complex

DEFINITION: Let M be a complex manifold, and (L, ∇) a flat, oriented, real line bundle. Identifying sections of L with automorphic forms of weight 1 on $\tilde{M}$ as above, we consider **the Hodge decomposition** $d_\theta = \partial_\theta + \overline{\partial}_\theta$, where d_θ is the de Rham differential on automorphic forms, and ∂_θ, $\overline{\partial}_\theta$ its Hodge components.

PROPOSITION: Let (L, ∇) be a weight bundle on a complex manifold, φ its trivialization, and θ a connection form. Denote by $\tilde{M} \xrightarrow{\pi} M$ the universal covering, and let Φ be a non-zero parallel section of $\pi^* L$. Consider the equivalence

$$(\Lambda^*(M) \otimes L, d_\theta) \xrightarrow{\Xi} (\Lambda^*(M)_\lambda, d)$$

constructed above. **Then $\partial_\theta = \partial - \theta^{1,0}$ and $\overline{\partial}_\theta = \overline{\partial} - \theta^{0,1}$.**

Proof: The map Ξ is compatible with the Hodge decomposition. ■

COROLLARY: The following commutation relations are clear: $\{\partial_\theta, \overline{\partial}_\theta\} = \{\partial_\theta, \partial_\theta\} = \{\overline{\partial}_\theta, \overline{\partial}_\theta\} = 0 = \{d_\theta, d_\theta^c\}$, and $-2\sqrt{-1}\, \partial_\theta \overline{\partial}_\theta = d_\theta d_\theta^c$, where $d_\theta^c = I d_\theta I^{-1} = -\sqrt{-1}\,(\partial_\theta - \overline{\partial}_\theta) = d^c - I(\theta)$. ■

Lee class of an LCK manifold

DEFINITION: Let (M,ω,θ) be an LCK manifold. The cohomology class $[\theta]\in H^1(M,\mathbb{R})$ of its Lee form θ is called **the Lee class** of M.

EXAMPLE: Let $M=C(S)/\mathbb{Z}$ be a Vaisman manifold, $[\theta]$ its Lee class. **Then $\lambda[\theta]$ is also a Lee class of an LCK structure, for any $\lambda>0$.**

Proof: Let $\varphi=t^2$ be an automorphic Kähler potential on $C(S)=S\times\mathbb{R}^{>0}$. Then $dd^c\log\varphi$ is semipositive, and its zero eigenspace is generated by $\frac{d}{dt}$. Then

$$dd^c\varphi^\alpha=dd^c e^{\alpha\log\varphi}=\alpha\varphi^\alpha dd^c\log\varphi+\alpha^2\varphi^\alpha d\log\varphi\wedge d^c\log\varphi$$

and this (1,1)-form is strictly positive for any $\alpha>0$. Its Lee form is αdt. ■

EXAMPLE: Consider an LCK manifold (M,ω,θ) with potential $\varphi\in C^\infty M$. **Then $a[\theta]$ is also a Lee class of an LCK structure, for any $a>1$.**

Proof: $dd^c\varphi^a=\varphi^{a-2}(a\cdot\varphi dd^c\varphi+a(a-1)d\varphi\wedge d^c\varphi)$, hence φ^a is also an automorphic potential, for any $a>1$. Its Lee form is $\varphi^{-a}d\varphi^a=a\theta$. ■

CONJECTURE: Let (M,ω,θ) be a compact LCK manifold such that $\lambda[\theta]$ is also a Lee class of an LCK structure, for any $\lambda>0$ (or $\lambda>1$). Then it admits a structure of a Vaisman manifold (or LCK manifold with potential).

Monodromy group and the Lee class

REMARK: **Monodromy group** of an LCK manifold (M, ω, θ) is defined as the Galois group of the smallest covering $\pi : \tilde{M} \longrightarrow M$ such that $\pi^*\theta$ is exact. **Rank** of an LCK manifold is rank of its monodromy group.

PROPOSITION: Let (M, ω, θ) be an LCK manifold and $[\theta]$ its Lee class. Consider a smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contains $[\theta]$. **Then $\dim V$ is equal to the rank of M.**

Proof: The group Γ is identified with an image of $\pi_1(M)$ under the map $[\theta] : \pi_1(M) \longrightarrow \mathbb{R}$, because it is equal to the monodromy of the weight bundle, and the monodromy along a loop γ is equal to $\int_\gamma \theta$. ■

Morse-Novikov class of an LCK manifold

DEFINITION: Let (M,ω,θ) be an LCK manifold, $d\omega = \omega\wedge\theta$. Then $d_\theta(\omega) = 0$. The cohomology class $[\omega]_{MN}$ of ω in the Morse-Novikov cohomology is called **Morse-Novikov class** of M.

CLAIM: **$[\omega]_{MN}$ vanishes for LCK manifold with potential and, hence, for Vaisman manifolds.**

Proof: Indeed, the corresponding automorphic form $\tilde\omega = \Xi(\omega)$ is a differential of an automorphic form, and the Morse-Novikov cohomology is cohomology of the complex of automorphic forms. ■

REMARK: **$[\omega]_{MN}$ is known to be non-zero for some other LCK manifolds.** All known examples of compact LCK manifolds with vanishing Morse-Novikov class admit an LCK metric with potential.

Bott-Chern cohomology

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{BC}(M)$ the space of closed (p,q)-forms modulo $dd^c(\Lambda^{p-1,q-1}(M))$. Then $H^{p,q}_{BC}(M)$ is called **the Bott-Chern cohomology** of M.

REMARK: There are natural (and functorial) maps from the Bott-Chern cohomology to the Dolbeault cohomology $H^*(\Lambda^{*,*}(M), \overline{\partial})$ and to the de Rham cohomology, but no morphisms between de Rham and Dolbeault cohomology.

THEOREM: Let M be a compact complex manifold. **Then $H^{p,q}_{BC}(M)$ is finite-dimensional.**

Proof: See below.

Differential operators

DEFINITION: (Grothendieck)
Let R be a commutative ring over a field k, and A, B R-modules. **Differential operator of order 0** from A to B is an R-linear map $\varphi \in \operatorname{Hom}_R(A, B)$. Differential operator of order $i > 0$ is defined inductively: $\alpha \in \operatorname{Diff}^i(A, B)$ if for any $r \in \mathbb{R}$, the commutator $\alpha L_r - L_r\alpha$ belongs to $\operatorname{Diff}^{i-1}(A, B)$, where $L_r(x) = rx$.

DEFINITION: Given a vector bundle on a smooth manifold M, we may consider its space of sections as an $C^\infty M$-module. **Differential operators** $\operatorname{Diff}^i(F, G)$ on vector bundles F, G are defined as differential operators on the corresponding spaces of sections in the sense of the Grothendieck's definition. **Differential operator on M** is an element of $\operatorname{Diff}^i(M) := \operatorname{Diff}^i(C^\infty M, C^\infty M)$.

REMARK: This definition is equivalent to the usual one: locally (in coordinates) any differential operator is expressed as a composition of derivations and multiplications by $f \in C^\infty M$.

Symbols

THEOREM: Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ....$ Then **its associated graded ring is isomorphic to $\bigoplus_i \operatorname{Sym}^i(TM)$,** identified with the ring if fiberwise polynomial functions on T^*M.

COROLLARY: Let F, G be vector bundles, and $\operatorname{Diff}^0(F,G) \subset \operatorname{Diff}^1(F,G) \subset \operatorname{Diff}^2(F,G)$ the corresponding spaces of differential operators. **Then**

$$\operatorname{Diff}^i(F,G)/\operatorname{Diff}^{i-1}(F,G) = \operatorname{Sym}^i(TM) \otimes \operatorname{Hom}(F,G),$$

where Sym^i denotes the symmetric power (symmetric part of the tensor power).

DEFINITION: Let F, G be vector bundles, and $D \in \operatorname{Diff}^i(F,G)$ a differential operator. Consider its class in $\operatorname{Diff}^i(F,G)/\operatorname{Diff}^{i-1}(F,G)$ as a $\operatorname{Hom}(F,G)$-valued function on $T^*(M)$ (polynomial of order i on each cotangent space). This function is called **symbol** of D.

EXERCISE: Let $D: B \longrightarrow B \otimes \Lambda^1 M$ be a first order differential operator. **Prove that D is a connection if and only if its symbol is equal to the identity operator** $\operatorname{Id} \in \operatorname{Hom}(\Lambda^1 M \otimes (\operatorname{Hom}(B, B \otimes \Lambda^1 M))$

EXERCISE: Prove that the symbol of the Laplacian operator $\Delta: \Lambda^* M \longrightarrow \Lambda^* M$ on a Riemannian manifold M at $\xi \in T^*M$ **is equal to $|\xi|^2 \operatorname{Id}_{\Lambda^* M}$.**

Elliptic operators

DEFINITION: Let F, G be vector bundles of the same rank. A differential operator $D: F \longrightarrow G$ is called **elliptic** if its symbol $\sigma(D) \in \mathrm{Hom}(F,G) \otimes \mathrm{Sym}^i(TM)$ is invertible at each non-zero $\xi \in T^*M$.

DEFINITION: Let F be a vector bundle on a compact manifold. The **L^2_p-topology** on the space of sections of F is a topology defined by a quadratic form $|f|^2 = \sum_{i=0}^p \int_M |\nabla^i f|^2$, for some connection and scalar product on F and $\Lambda^1 M$.

EXERCISE: Prove that this topology is independent from the choice of a connection and a metric.

DEFINITION: A continuous operator $\psi: A \longrightarrow B$ on topological vector spaces is called **Fredholm** if its kernel is finite-dimensional, and its image is closed, and has finite codimension.

THEOREM: Let $D: F \longrightarrow G$ be an elliptic operator of order d. Clearly, D defines a continuous map $L^2_p(F) \longrightarrow L^2_{p-d}(G)$. **Then this map is Fredholm.**

REMARK: This difficult theorem is a foundation of Hodge theory (and many other things besides).

Elliptic complexes

DEFINITION: Let F, G, H be vector bundles, and $F \xrightarrow{D} G \xrightarrow{D} H$ a complex of differential operators (that is, $D^2 = 0$). It is called **elliptic complex** if its symbols $F \xrightarrow{\sigma(D)} G \xrightarrow{\sigma(D)} H$ give an exact sequence at each non-zero $\xi \in T^*M$.

DEFINITION: Let A, B, C be topological vector spaces and $A \xrightarrow{D} B \xrightarrow{D} C$ a complex of continuous maps. It is called **Fredholm complex** if $\operatorname{im} D$ is closed, and $\frac{\ker D}{\operatorname{im} D}$ is finite-dimensional.

THEOREM: Let $F \xrightarrow{D_1} G \xrightarrow{D_2} H$ be an elliptic complex of differential operators, with D_1 of order d_1 and D_2 of order d_2. **Then the complex $L^2_p(F) \xrightarrow{D_1} L^2_{p-d_1}(G) \xrightarrow{D_2} L^2_{p-d_1-d_2}(H)$ is Fredholm.**

COROLLARY: Cohomology of any elliptic complex are finite-dimensional.

Bott-Chern cohomology are finite-dimensional

Now we can prove

THEOREM: Let M be a compact complex manifold. **Then $H^{p,q}_{BC}(M)$ is finite-dimensional.**

Proof: It would suffice to show that the complex

$$\Lambda^{p-1,q-1}(M) \xrightarrow{dd^c} \Lambda^{p,q}(M) \xrightarrow{\partial+\overline{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M)$$

is elliptic. **At $\xi \in T^*M = \Lambda^{1,0}(M)$, symbol of dd^c is equal to multiplication of a form by $\xi\wedge\overline{\xi}$, the symbol of ∂ is multiplication by ξ and the symbol of $\overline{\partial}$ is multiplication by $\overline{\xi}$.** Therefore, $\ker\sigma(\partial) = \operatorname{im}\sigma(\partial)$, $\ker\sigma(\overline{\partial}) = \operatorname{im}\sigma(\overline{\partial})$ (this proves finite-dimensionality of Dolbeault cohomology), and $\ker\sigma(\overline{\partial}) \cap \ker\sigma(\partial) = \operatorname{im}\sigma(\partial\overline{\partial})$. ■

Weighted Bott-Chern cohomology

DEFINITION: Let M be a complex manifold, and L a flat vector bundle. Consider the corresponding differential $d_\nabla = d_\theta$, and let ∂_θ, $\overline{\partial}_\theta$ be its Hodge components. **The weighted Bott-Chern cohomology** are defined as

$$H^{p,q}_{BC}(M,L) := \frac{\ker d_\theta\big|_{\Lambda^{p,q}(M)\otimes L}}{\operatorname{im}\partial_\theta\overline{\partial}_\theta}.$$

THEOREM: Let M be a compact complex manifold, and L a flat vector bundle. **Then the group $H^{p,q}_{BC}(M,L)$ is finite-dimensional.**

Proof: The complex

$$\Lambda^{p-1,q-1}(M)\otimes L \xrightarrow{d_\theta d^c_\theta} \Lambda^{p,q}(M)\otimes L \xrightarrow{\partial_\theta+\overline{\partial}_\theta} \Lambda^{p+1,q}(M)\oplus\Lambda^{p,q+1}(M)\otimes L$$

is equal to the usual Bott-Chern complex up to terms of lower order, hence it has the same symbols. ■

Bott-Chern class

DEFINITION: (M, ω, θ) be an LCK manifold, and L its weight bundle. The cohomology class of ω in $H^{1,1}_{BC}(M, L)$ is called **Bott-Chern class of M**.

REMARK: It is the best analogue of the Kähler class, and the following theorem (together with the Hopf embedding resukt) an LCK analogue of Kodaira embedding theorem.

THEOREM: Let (M, ω, θ) be an LCK manifold. Suppose that its Lee class $[\theta]$ is proportional to a rational class in $H^1(M)$ and $[\omega]_{BC} = 0$. **Then (M, ω, θ) is an LCK manifold with potential.**

Proof: Existence of an automorphic potential is precisely vanishing of $[\omega]_{BC} = 0$. Its properness is equivalent to $\Gamma \cong \mathbb{Z}$, where Γ is a monodromy group of M. Since rank of Γ is equal to the dimension of a smallest rational subspace generated by $[\theta]$, it is equal 1. ■

Open questions

A weighted version of dd^c-lemma is known to be wrong, even for Vaisman manifolds (Goto). However, the following (very weak) version of $d_\theta d_\theta^c$-lemma could be true.

PROBLEM: Let M be a compact LCK manifold with its Morse-Novikov class $[\omega]_{MN}$ equal zero. **Would it follow that M has monodromy $\mathbb{Z}$? Would it follow that M admits an LCK metric with potential, when its monodromy is $\mathbb{Z}$?**

PROBLEM: Find an example of locally (but not globally) conformally symplectic manifold of dimension $\geqslant 3$ not admitting LCK structure.

PROBLEM: Prove that a compact torus with non-Kähler complex structure does not admit an LCK metric, or find one.