Locally conformally Kähler manifolds

lecture 13: automorphisms of LCK manifolds

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Local systems (reminder)

DEFINITION: A **local system** is a locally constant sheaf of vector spaces.

THEOREM: A local system with fiber B at $x \in M$ gives a homomorphism $\pi_1(M, x) \longrightarrow \operatorname{Aut}(B)$. **This correspondence gives an equivalence of categories.**

DEFINITION: A bundle (B, ∇) is called **flat** if its curvature vanishes.

DEFINITION: A section b of (B, ∇) is called **parallel** if $\nabla(b) = 0$.

CLAIM: Let (B, ∇) be a flat bundle on M, and $\mathcal{B}$ be the sheaf of parallel sections. **Then $\mathcal{B}$ is a locally constant sheaf.**

THEOREM: This correspondence **gives an equivalence of categories** of flat bundles and local systems.

LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

LCK manifolds with potential (reminder)

DEFINITION: Let M be an LCK manifold, and $(\tilde M, \tilde\omega)$ its Kähler covering. It is called **LCK manifold with potential** if $\tilde M$ admits an automorphic Kähler potential $\varphi: \tilde M \longrightarrow \mathbb{R}^{>0}$, $dd^c\varphi = \tilde\omega$, which is **proper** (preimage of a compact is again compact).

THEOREM: The property of being LCK with potential is stable under small deformations.

THEOREM: Let M be an LCK manifold, $\Gamma \subset \mathbb{R}^{>0}$ the monodromy group, and $(\tilde M, \tilde\omega)$ its Kähler covering, with $\tilde M/\Gamma = M$. Assume that $\tilde\omega$ admits a Γ-automorphic Kähler potential φ. **The map φ is proper if and only if $\Gamma = \mathbb{Z}$.**

THEOREM: Let M be an LCK manifold with potential, and $\tilde M$ its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde M_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde M \subset \tilde M_c$. Moreover, the monodromy action on $\tilde M$ is extended to a holomorphic automorphism of $\tilde M_c$.

THEOREM: Let M be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$. **Then M admits a holomorphic embedding to a linear Hopf manifold.**

Morse-Novikov cohomology (reminder)

DEFINITION: Define **the B-valued de Rham differential** $d_\nabla : \Lambda^i(M) \otimes B \longrightarrow \Lambda^{i+1}(M) \otimes B$ as $d_\nabla(\eta \otimes b) := d\eta \otimes b + (-1)^{\tilde\eta} \eta \wedge \nabla b$. **It is easy to check that $d_\nabla^2 = 0$ if and only if the curvature of ∇ vanishes.**

CLAIM: The cohomology of the complex $(\Lambda^* M \otimes B, d_\nabla)$ are equal to the cohomology of the local system $\mathcal{B} := \ker \nabla$.

REMARK: Let B be a line bundle equipped with a flat connection, φ its trivialization, and θ its connection form, $\nabla(f\varphi) = df \otimes \varphi + f\theta \otimes \psi$. Then $d_\nabla(\eta \otimes \psi) = d\eta \otimes \psi + \theta \wedge \eta \otimes \psi$. This is written as $d_\nabla = d + \theta$.

DEFINITION: Cohomology of the complex $(\Lambda^* M, d_\theta := d - \theta)$ are called **Morse-Novikov cohomology**, or **Lichnerowicz cohomology**; the corresponding complex – **Morse-Novikov complex**. They compute the cohomology of the local system L^{-1}

Automorphic forms (reminder)

DEFINITION: Let M be a manifold, $\tilde{M}$ its Galois covering. A form η on $\tilde{M}$ is called **automorphic** if for any $\gamma \in \pi_1(M)$ acting on $\tilde{M}$ as usual, the form $\gamma^*\eta$ is proportional to η. The character $\chi_\eta(\gamma) := \frac{\gamma^*\tilde{\eta}}{\tilde{\eta}}$ is called **the character of automorphy**, or **weight** for η.

DEFINITION: Let L be an oriented real line bundle equipped with a flat connection (we call L **weight bundle**), and $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ its monodromy. **Automorphic form of weight λ** is an automorphic form which satisfies $\gamma^*\tilde{\eta} = \lambda^{-1}\chi(\gamma)\eta$ for each $\gamma \in \pi_1(M)$. We denote the space of such forms by $\Lambda^*(M)_\lambda$.

PROPOSITION: The Morse-Novikov complex with coefficients in L^λ is identified with the de Rham complex of automorphic forms of weight λ on $\tilde{M}$.

Morse-Novikov Dolbeault complex (reminder)

DEFINITION: Let M be a complex manifold, and (L,∇) a flat, oriented, real line bundle. Identifying sections of L with automorphic forms of weight 1 on $\tilde M$ as above, we consider **the Hodge decomposition $d_\theta = \partial_\theta + \overline{\partial}_\theta$,** where d_θ is the de Rham differential on automorphic forms, and ∂_θ, $\overline{\partial}_\theta$ its Hodge components.

PROPOSITION: Let (L,∇) be a weight bundle on a complex manifold, φ its trivialization, and θ a connection form. Denote by $\tilde M \xrightarrow{\pi} M$ the universal covering, and let Φ be a non-zero parallel section of $\pi^* L$. Consider the equivalence

$$(\Lambda^*(M)\otimes L, d_\theta) \xrightarrow{\Xi} (\Lambda^*(M)_\lambda, d)$$

between Morse-Novikov complex and the de Rham complex of automorphic forms. Then **this identification is compatible with Dolbeault decomposition, and gives an equivalence between ∂_θ, $\overline{\partial}_\theta$ and Dolbeault differentials on $\Lambda^*(M)_\lambda$.**

COROLLARY: This gives the following commutation relations: $\{\partial_\theta, \overline{\partial}_\theta\} = \{\partial_\theta, \partial_\theta\} = \{\overline{\partial}_\theta, \overline{\partial}_\theta\} = 0 = \{d_\theta, d_\theta^c\}$, and $-2\sqrt{-1}\,\partial_\theta\overline{\partial}_\theta = d_\theta d_\theta^c$, where $d_\theta^c = I d_\theta I^{-1} = -\sqrt{-1}\,(\partial_\theta - \overline{\partial}_\theta) = d^c - I(\theta)$. ∎

Lee class of an LCK manifold (reminder)

DEFINITION: Let (M,ω,θ) be an LCK manifold. The cohomology class $[\theta]\in H^1(M,\mathbb{R})$ of its Lee form θ is called **the Lee class** of M.

REMARK: Monodromy group of an LCK manifold (M,ω,θ) is defined as the Galois group of the smallest covering $\pi: \tilde M \longrightarrow M$ such that $\pi^*\theta$ is exact. **Rank** of an LCK manifold is rank of its monodromy group.

PROPOSITION: Let (M,ω,θ) be an LCK manifold and $[\theta]$ its Lee class. Consider a smallest rational subspace $V\subset H^1(M,\mathbb{Q})$ such that $V\otimes_{\mathbb{Q}}\mathbb{R}$ contains $[\theta]$. **Then $\dim V$ is equal to the rank of M.**

Proof: The group Γ is identified with an image of $\pi_1(M)$ under the map $[\theta]: \pi_1(M)\longrightarrow\mathbb{R}$, because it is equal to the monodromy of the weight bundle, and the monodromy along a loop γ is equal to $e^{\int_\gamma\theta}$. ■

Morse-Novikov class of an LCK manifold (reminder)

DEFINITION: Let (M, ω, θ) be an LCK manifold, $d\omega = \omega \wedge \theta$. Then $d_\theta(\omega) = 0$. The cohomology class $[\omega]_{MN}$ of ω in the Morse-Novikov cohomology is called **Morse-Novikov class** of M.

CLAIM: $[\omega]_{MN}$ **vanishes for LCK manifolds with potential and, hence, for Vaisman manifolds.**

Proof: Indeed, the corresponding automorphic form $\tilde{\omega} = \Xi(\omega)$ is a differential of an automorphic form, and the Morse-Novikov cohomology is cohomology of the complex of automorphic forms. ■

REMARK: $[\omega]_{MN}$ **is known to be non-zero for some other LCK manifolds.** All known examples of compact LCK manifolds with vanishing Morse-Novikov class admit an LCK metric with potential.

Bott-Chern cohomology (reminder)

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{BC}(M)$ the space of closed (p,q)-forms modulo $dd^c(\Lambda^{p-1,q-1}(M))$. Then $H^{p,q}_{BC}(M)$ is called **the Bott-Chern cohomology** of M.

THEOREM: Let M be a compact complex manifold. **Then $H^{p,q}_{BC}(M)$ is finite-dimensional.**

DEFINITION: Let M be a complex manifold, and L a flat vector bundle. Consider the corresponding differential $d_\nabla = d_\theta$, and let ∂_θ, $\bar\partial_\theta$ be its Hodge components. **The weighted Bott-Chern cohomology** are defined as

$$H^{p,q}_{BC}(M,L) := \frac{\ker d_\theta\Big|_{\Lambda^{p,q}(M)\otimes L}}{\operatorname{im} \partial_\theta\bar\partial_\theta}.$$

THEOREM: Let M be a compact complex manifold, and L a flat vector bundle. **Then the group $H^{p,q}_{BC}(M,L)$ is finite-dimensional.**

Bott-Chern class (reminder)

DEFINITION: (M, ω, θ) be an LCK manifold, and L its weight bundle. The cohomology class of ω in $H^{1,1}_{BC}(M, L)$ is called **Bott-Chern class of M**.

REMARK: It is the best analogue of the Kähler class, and the following theorem (together with the Hopf embedding result) is an LCK analogue of Kodaira embedding theorem.

THEOREM: Let (M, ω, θ) be an LCK manifold. Suppose that its Lee class $[\theta]$ is proportional to a rational class in $H^1(M)$ and $[\omega]_{BC} = 0$. **Then (M, ω, θ) is an LCK manifold with potential.**

Proof: Existence of an automorphic potential is precisely vanishing of $[\omega]_{BC}$. Its properness is equivalent to $\Gamma \cong \mathbb{Z}$, where Γ is a monodromy group of M. Since rank of Γ is equal to the dimension of a smallest rational subspace generated by $[\theta]$, it is equal 1. ■

Open questions (reminder)

A weighted version of dd^c-lemma is known to be wrong, even for Vaisman manifolds (Goto). However, the following (very weak) version of $d_\theta d^c_\theta$-lemma could be true.

PROBLEM: Let M be a compact LCK manifold with its Morse-Novikov class $[\omega]_{MN}$ equal zero. **Would it follow that M has monodromy $\mathbb{Z}$? Would it follow that M admits an LCK metric with potential, when its monodromy is $\mathbb{Z}$?**

PROBLEM: Find an example of locally (but not globally) conformally symplectic manifold of dimension $\geqslant 3$ not admitting LCK structure.

PROBLEM: Prove that a compact torus with non-Kähler complex structure does not admit an LCK metric, or find one.

LCK manifolds with S^1-action: main theorem

THEOREM: Let M be a compact complex manifold, equipped with a holomorphic S^1-action and an LCK metric (not necessarily compatible). Suppose that the weight bundle L, restricted to a general orbit of this S^1-action, is non-trivial as a 1-dimensional local system. **Then M admits an LCK metric with an automorphic potential.**

The proof takes the rest of this lecture.

REMARK: The converse statement is also true. Indeed, let $M = \tilde{M}/\mathbb{Z}$ be an LCK manifold with potential, $\tilde{M}$ be its Kähler covering. As we have already shown, the $\mathbb{Z}$-action on $\tilde{M}$ admits a logarithm, given by a holomorphic vector field $A \in T\tilde{M}$. **Then e^{tA} is a holomorphic S^1-action with the required properties.**

LCK manifolds with S^1-action: Lee form

REMARK: **Conformally equivalent metrics** are metrics $g, g' = e^f g$. **Conformal class** of a metric is its class of conformal equivalence.

LEMMA: Let M be a compact complex manifold, equipped with a holomorphic S^1-action and an LCK metric (not necessarily compatible). **Then there exists an LCK metric in the same conformal class with S^1-invariant Lee form.**

Proof: Let G be a compact subgroup of Aut(M). Averaging the Lee form θ on G, we obtain a closed 1-form θ' which is S^1-invariant and stays in the same cohomology class as θ: $\theta' = \theta + df$. Then $\omega' = e^{-f}\omega$ is a LCK form with Lee form θ' and conformal to ω. ■

LCK manifolds with S^1-action: S^1-invariance

Proposition 1: Let (M, ω, θ) be a compact complex manifold, equipped with a holomorphic S^1-action and an LCK metric (not necessarily compatible). **Then M admits an S^1-invariant LCK metric.**

Proof. Step 1: Using the previous lemma, we chose a metric in the same conformal class with S^1-invariant Lee form. **Therefore, we may assume θ is S^1-invariant.**

Proof. Step 2: For each $t \in S^1$, let $\omega_t := \rho(t)^*\omega$. Then $d(\omega_t) = \omega_t \wedge \theta$. Averaging ω_t with respect to t, **we obtain a positive, S^1-invariant form ω_{av} satisfying $d(\omega_{av}) = \omega_{av} \wedge \theta$.** ■

CLAIM: Let (M, ω, θ) be a compact complex manifold, equipped with a holomorphic S^1-action by isometries. Then, on a Kähler covering $\tilde{M} \longrightarrow M$, **the corresponding action of the covering $\tilde{S}^1 = \mathbb{R}$ is by holomorphic homotheties.**

Proof: Indeed, if two Kähler forms are conformally equivalent, they are proportional. ■

Holomorphic homotheties on Kähler manifolds

Proposition 2: Let A be a vector field acting on a Kähler manifold $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties: $\operatorname{Lie}_A \tilde{\omega} = \lambda\tilde{\omega}$, with $\lambda \neq 0$. **Then**

$$dd^c|A|^2 = \lambda^2\tilde{\omega} + \operatorname{Lie}^2_{A^c}\tilde{\omega},$$

where $A^c = I(A)$**.**

Proof. Step 1: Let $\eta := \tilde{\omega} \lrcorner A = I(A)^\flat$ be the dual form to A^c. Replacing A by $\lambda^{-1}A$, we may assume that $\lambda = 1$. **By Cartan's formula,**

$$\tilde{\omega} = \operatorname{Lie}_A \tilde{\omega} = d(\tilde{\omega} \lrcorner A) = d\eta.$$

Step 2: Since A and A^c are holomorphic, Lie_{A^c} commutes with I. This gives

$$\operatorname{Lie}_{A^c}\tilde{\omega} = \operatorname{Lie}_{A^c} I\tilde{\omega} = I \operatorname{Lie}_{A^c}\tilde{\omega} = IdI^{-1}(\tilde{\omega} \lrcorner A) = d^c\eta.$$

Step 3: Since Lie_A commutes with I, **one has** $\{d^c, i_{A^c}\} = I \operatorname{Lie}_A I^{-1} = \operatorname{Lie}_A$**,** where $i_v(\alpha) = \alpha \lrcorner v$ is the contraction operator.

Holomorphic homotheties on Kähler manifolds (2)

Proposition 2: Let A be a vector field acting on a Kähler manifold $(\tilde M, \tilde\omega)$ by holomorphic homotheties: $\operatorname{Lie}_A \tilde\omega = \lambda\tilde\omega$, with $\lambda \neq 0$. **Then**

$$dd^c|A|^2 = \lambda^2\tilde\omega + \operatorname{Lie}^2_{A^c}\tilde\omega,$$

where $A^c = I(A)$.

Proof. Step 1: Let $\eta := \tilde\omega \lrcorner A$, and $\lambda = 1$. Then $\tilde\omega = \operatorname{Lie}_A \tilde\omega = d(\tilde\omega \lrcorner A) = d\eta$.

Step 2: $\operatorname{Lie}_{A^c}\tilde\omega = \operatorname{Lie}_{A^c} I\tilde\omega = I \operatorname{Lie}_{A^c}\tilde\omega = IdI^{-1}(\tilde\omega \lrcorner A) = d^c\eta$.

Step 3: $\{d^c, i_{A^c}\} = I \operatorname{Lie}_A I^{-1} = \operatorname{Lie}_A$, where $i_v(\alpha) = \alpha \lrcorner v$ is the contraction operator.

Step 4:

$$\operatorname{Lie}^2_{A^c}\tilde\omega = \operatorname{Lie}_{A^c} d^c\eta = i_{A^c}dd^c\eta + di_{A^c}d^c\eta \qquad (*)$$

(Step 2 and Cartan's formula). The first summand vanishes because $dd^c\eta = -d^c d\eta = d^c\tilde\omega$ (Step 1). The second summand gives

$$di_{A^c}d^c\eta = dd^c\langle I(A), I(A)^\flat\rangle - d\{d^c, i_{A^c}\}\eta \qquad (**)$$

Finally, $d\{d^c, i_{A^c}\}\eta = d\operatorname{Lie}_A \eta = \operatorname{Lie}_A d\eta = \tilde\omega$ (Step 3 and Step 1). Therefore, (*) and (**) give $\operatorname{Lie}^2_{A^c}\tilde\omega = dd^c|A|^2 - \omega$, for $\lambda = 1$. ■

LCK manifolds with S^1-action: main theorem (proof)

THEOREM: Let M be a compact complex manifold, equipped with a holomorphic S^1-action ρ and an LCK metric (not necessarily compatible). Suppose that the weight bundle L, restricted to a general orbit of this S^1-action, is non-trivial as a 1-dimensional local system. **Then M admits an LCK metric with an automorphic potential.**

Proof. Step 1: Using Proposition 1, **we may assume that the metric ω on M is S^1-invariant.** Denote the corresponding Kähler metric on $\tilde M$ by $\tilde\omega$, and let $\tilde\rho$ be the lift of S^1-action to $\tilde M$. Since conformally equivalent Kähler metrics are proportional, **$\tilde\rho$ acts by homotheties.**

Step 2: Restriction of the flat connection in the weight bundle L to a loop has trivial monodromy whenever this loop lifts to a homeomorphic loop in $\tilde M$. Since L is non-trivial on orbits of ρ, the lift $\tilde\rho$ is an $\mathbb R$-action, not reducible to S^1-action. Denote the kernel of the natural map $\operatorname{im}\tilde\rho \longrightarrow \operatorname{im}\rho$ by Γ. Since $\tilde M$ is a minimal Kähler covering, Γ acts on $\tilde M$ non-isometrically, hence $\tilde\rho$ acts by non-trivial homotheties. Rescaling, **we may assume that the vector field A tangent to $\tilde\rho$ satisfies $\operatorname{Lie}_A\tilde\omega = \tilde\omega$.**

LCK manifolds with S^1-action: main theorem (proof, part 2)

Step 3: Proposition 2 gives

$$\tilde\omega = dd^c|A|^2 - \operatorname{Lie}^2_{A^c}\tilde\omega, \qquad (***)$$

where A is the homothety vector field tangent to $\tilde\rho$, and $A^c = I(A)$. Let $\mu_t := \rho^c(t)^*[\omega]_{BC}$ be the Bott-Chern class of $e^{tA^c}(\omega)$. By (***), μ_t satisfies the differential equation $\mu_t'' = -\mu_t$, hence **$\mu_t = a\sin(t) + b\cos(t)$, for some $a, b \in H^{1,1}_{BC}(M, L)$.**

Step 4: From Step 3 it follows that $\int_0^{2\pi} e^{tA^c}[\tilde\omega]dt = 0$. Consider the Kähler form $\tilde\omega_W := \int_0^{2\pi} e^{tA^c}(\tilde\omega)dt = 0$ on $\tilde M$. This form is an average of automorphic forms of the same character of automorphicity, because e^{tA^c} commutes with $e^{t'A}$. The Bott-Chern class of ω_W vanishes, because $\int_0^{2\pi}\sin(t)dt = \int_0^{2\pi}\cos(t)dt = 0$. **Therefore, $\tilde\omega_W$ admits an automorphic potential.**

Step 5: To finish the proof, **it remains to show that the monodromy of M is $\mathbb{Z}$.** This is implied by the theorem proven in Lecture 5.

THEOREM: Let (M, ω, θ) be a compact LCK manifold, and X a vector field acting on M by isometries and on $\tilde M$ by non-isometric homotheties. **Then $\operatorname{Mon}(M) = \mathbb{Z}$.** ■