Locally conformally Kähler manifolds

lecture 14: Oeljeklaus-Toma manifolds

Misha Verbitsky

HSE and IUM, Moscow

June 2, 2014

Local systems (reminder)

DEFINITION: A **local system** is a locally constant sheaf of vector spaces.

THEOREM: A local system with fiber B at $x \in M$ gives a homomorphism $\pi_1(M, x) \longrightarrow \operatorname{Aut}(B)$. **This correspondence gives an equivalence of categories.**

DEFINITION: A bundle (B, ∇) is called **flat** if its curvature vanishes.

DEFINITION: A section b of (B, ∇) is called **parallel** if $\nabla(b) = 0$.

CLAIM: Let (B, ∇) be a flat bundle on M, and $\mathcal{B}$ be the sheaf of parallel sections. **Then $\mathcal{B}$ is a locally constant sheaf.**

THEOREM: This correspondence **gives an equivalence of categories** of flat bundles and local systems.

LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Lee class of an LCK manifold (reminder)

DEFINITION: Let (M,ω,θ) be an LCK manifold. The cohomology class $[\theta]\in H^1(M,\mathbb{R})$ of its Lee form θ is called **the Lee class** of M.

REMARK: Monodromy group of an LCK manifold (M,ω,θ) is defined as the Galois group of the smallest covering $\pi:\ \tilde M \longrightarrow M$ such that $\pi^*\theta$ is exact. **Rank** of an LCK manifold is rank of its monodromy group.

PROPOSITION: Let (M,ω,θ) be an LCK manifold and $[\theta]$ its Lee class. Consider a smallest rational subspace $V\subset H^1(M,\mathbb{Q})$ such that $V\otimes_{\mathbb{Q}}\mathbb{R}$ contains $[\theta]$. **Then $\dim V$ is equal to the rank of M.**

Proof: The group Γ is identified with an image of $\pi_1(M)$ under the map $[\theta]:\ \pi_1(M)\longrightarrow \mathbb{R}$, because it is equal to the monodromy of the weight bundle, and the monodromy along a loop γ is equal to $e^{\int_\gamma\theta}$. ■

Solvmanifolds

DEFINITION: Let M be a smooth manifold equipped with a transitive action of solvable Lie group G. Then M is called **a solvmanifold**. If G is nilpotent, M is called **a nilmanifold**.

REMARK: All solvmanifolds are obtained as quotient spaces, $M = G/H$ (Mostow). All nilmanifolds are obtained as quotient spaces $M = G/\Gamma$, where Γ is discrete (Maltsev).

DEFINITION: **An integrable complex structure** on a real Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$

REMARK: Right-invariant complex structures on a connected real Lie group **are in 1 to 1 correspondence with integrable complex structures** on its Lie algebra.

DEFINITION: A **complex solvmanifold** is a solvmanifold $M = G/H$ equipped with a complex structure, in such a way that G has a right-invariant complex structure, and the projection $G \longrightarrow M$ is holomorphic.

REMARK: **Solvmanifolds are usually non-homogeneous** (as complex manifolds).

Normed fields

DEFINITION: An absolute value on a field k is a function $|\cdot|: k \longrightarrow \mathbb{R}^{\geqslant 0}$, satisfying the following

1. **Zero:** $|x| = 0 \Leftrightarrow x = 0$.

2. **Multiplicativity:** $|xy| = |x||y|$.

3. **There exists $c > 0$ such that $|\cdot|^c$ satisfies the triangle inequality.**

EXAMPLE: The usual absolute value on $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

EXAMPLE: Let p – be a prime number, and $m, n \in \mathbb{Z}$ coprime with p. Define **p-adic absolute value** on $\mathbb{Q}$ via $|\frac{m}{n}p^k| := p^{-k}$.

REMARK: p-adic absolute value satisfies an additional "non-archimedean axiom": $|x+y| \leqslant \max(|x|, |y|)$. Such absolute values are called **non-archimedean**.

REMARK: Any power of non-archimedean absolute value is again non-archimedean, and satisfies the triangle inequality.

Normed fields and topology

DEFINITION: Let $|\cdot|$ be an absolute value on a field F. Consider topology on F with open sets generated by

$$B_\varepsilon(x) := \{y \in k \ \mid \ |x-y| < \varepsilon\}.$$

Absolute values are called **equivalent** if they induce the same topology.

THEOREM: Absolute values $|\cdot|_1, |\cdot|_2$ are equivalent if and only if $|\cdot|_1 = |\cdot|_2^c$ for some $c > 0$.

THEOREM: (Ostrowski) **Every absolute value on $\mathbb{Q}$ is equivalent to the usual ("archimedean") one or to p-adic one.**

DEFINITION: A completion of a field k under an absolute value $|\cdot|$ is a completion of k in a metric $|\cdot|^c$, where $c > 0$ is a constant such that $|\cdot|^c$ satisfies the triangle inequality.

REMARK: A completion of a field is again a field.

EXAMPLE: A completion of $\mathbb{Q}$ under the p-adic absolute value is called **a field of p-adic numbers**, denoted $\mathbb{Q}_p$.

Local fields

DEFINITION: A finite extension $K:k$ of fields is a field $K \supset k$ which is finite-dimensional as a vector space over k. **A number field** is a finite extension of $\mathbb{Q}$. **Functional field** is a finite extension of $\mathbb{F}_p(t)$. **Global field** is a number or functional field. **Local field** is a completion of a global field under a non-trivial absolute value.

THEOREM: Let $\bar{k}$ be a field which is complete and locally compact under some absolute value. **Then $\bar{k}$ is a local field**.

DEFINITION: Let $K{:}k$ be a finite extension, and $x \in K$. Consider the multiplication by x as a k-linear endomorphism of K. Define **the norm** $N_{K/k}(x)$ as a determinant of this operator.

REMARK: Norm defines a homomorphism of multiplicative groups $K^* \longrightarrow k^*$.

REMARK: For Galois extensions, the norm $N_{K/k}(x)$ **is a product of all elements conjugate to x.**

THEOREM: Let $\overline{K}:\bar{k}$ be a finite extension of local fields, degree n. **Then an absolute value on $\bar{k}$ is uniquely extended to $\overline{K}$.** Moreover, **this extension is expressed as $|x| := \left|N_{K/k}(x)\right|^{\frac{1}{n}}$.**

Absolute values and extensions of global fields

CLAIM: Let A, B be extensions of a field k, $\operatorname{char} k = 0$, where $A{:}k$ is finite. Consider $A \otimes_k B$ as an k-algebra. **Then $A \otimes_k B$ is a direct sum of fields, containing A and B.**

THEOREM: Let k be a number field, $|\cdot|$ an absolute value, $K{:}k$ a finite extension, and $\overline{k}$ – its completion. Consider a decomposition $K \otimes_k \overline{k}$ into a direct sum of fields $K \otimes_k \overline{k} := \bigoplus_i \overline{K}_i$. **Then each extension of an absolute value $|\cdot|$ from k to K is induced from some $\overline{K}_i$, and all such extensions are non-equivalent.**

REMARK: When $k = \mathbb{Q}$, and $|\cdot|$ is the usual (archimedean) absolute value, we obtain that all K_i are extensions of $\mathbb{R}$, that is, isomorphic to $\mathbb{R}$ or $\mathbb{C}$. This gives

COROLLARY: For each number field K of degree n over $\mathbb{Q}$, there exists only a finite number of different homomorphisms $K \hookrightarrow \mathbb{C}$, all of them injective. Denote by s the number of embeddings whose image lies in $\mathbb{R} \subset \mathbb{C}$ (such an embedding is called **real**), and $2t$ the number of embedding, whose image does not lie in $\mathbb{R}$ ("**complex embeddings**). Then $s + 2t = n$.

Dirichlet unit theorem

DEFINITION: Let $K:\mathbb{Q}$ be a number field of degree n. **The ring of integers** $\mathcal{O}_K \subset K$ is an integral closure of $\mathbb{Z}$ in K, that is, the set of all roots in K of monic polynomials $P(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + ... + a_0$ with integer coefficients $a_i \in \mathbb{Z}$.

CLAIM: An additive group $\mathcal{O}_K^+$ is a finitely generated abelian group of rank n.

DEFINITION: A unit of a ring $\mathcal{O}_K$ is an element $u \in \mathcal{O}_K$, such that u^{-1} also belongs to $\mathcal{O}_K$.

REMARK: $u \in \mathcal{O}_K$ is a unit if and only if the norm $N_{K/\mathbb{Q}}(x) \in \mathbb{Z}$ is invertible, that is, $N_{K/\mathbb{Q}}(x) = \pm 1$.

Dirichlet's unit theorem: Let K be a number field which has s real embeddings and $2t$ complex ones. Then **the group of units $\mathcal{O}_K^*$ is isomorphic to $G \times \mathbb{Z}^{t+s-1}$,** where G is a finite group of roots of unity contained in K. Moreover, if $s > 0$, one has $G = \pm 1$.

REMARK: For a quadratic field, the group of units is a group of solutions of Pell's equation.

Cubic fields and complex surfaces

Let $K:\mathbb{Q}$ be a cubic extension of $\mathbb{Q}$ which has 2 complex embeddings τ, $\bar\tau$ and one real one σ (such an extension is obtained by adding all roots of a cubic polynomial which has exactly one real root).

REMARK: Due to Dirichlet theorem, $\mathcal{O}_K^*$ is isomorphic to $\mathbb{Z}\times\{\pm 1\}$. Let $\mathcal{O}_K^{*,+} := \sigma^{-1}(\mathbb{R}^{>0})\cap\mathcal{O}_K^*$. Then **the group $\mathcal{O}_K^{*,+}$ is isomorphic to $\mathbb{Z}$.**

Consider the action of $\mathcal{O}_K^+ \cong \mathbb{Z}^3$ on $\mathbb{R}^3 = \mathbb{C}\times\mathbb{R}$

$$\rho^+(x)(z,t) := (z+\tau(x), t+\sigma(x)).$$

Let Γ be a semidirect product $\mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,+}$, defined from the natural action of $\mathcal{O}_K^{*,+}$ on $\mathcal{O}_K^+$. **Define an action of Γ on $\mathbb{C}\times\mathbb{H}$, where $\mathbb{H}$ is an upper halfplane**, as follows.

The subgroup $\mathcal{O}_K^+\subset\Gamma$ acts on $\mathbb{C}\times\mathbb{H} = \mathbb{C}\times\mathbb{R}\times\mathbb{R}^{>0}$ by translations as above (trivially on the last argument), and $\mathcal{O}_K^{*,+}$ acts multiplicatively as

$$\rho^*(\xi)(z,z') := (\tau(\xi)z, \sigma(\xi)z').$$

Inoue surfaces of type S^0

DEFINITION: The **Inoue surface of type S^0** is a quotient $(\mathbb{C}\times\mathbb{H})/\Gamma$.

Its properties: 1. It is a compact, complex solvmanifold

2. Inoue surface **admits a flat connection preserving the complex structure** (by construction).

3. Its cohomology are the same as of $S^3\times S^1$

THEOREM: The Inoue surface $M := (\mathbb{C}\times\mathbb{H})/\Gamma$ **is locally conformally Kähler.**

Proof: Let z, u be coordinates on $\mathbb{C}\times\mathbb{H}$, and $\varphi(z,u) := |z|^2 + \operatorname{Im}(u)^{-1}$. Since $dd^c\varphi = \sqrt{-1}\, dz\wedge d\bar z + 2\sqrt{-1}\ \operatorname{Im}(u)^{-3} du\wedge du\bar u$, it is a Kähler form. Clearly, $dd^c\varphi$ is $\mathcal{O}_K^+$-invariant. Let $\varepsilon\in\mathbb{Z}=\mathcal{O}_K^{*,+}$ be a unit. Since $N(\varepsilon)$ is integer and invertible, $N(\varepsilon)=1$. This implies that $\tau(\varepsilon)^2=\sigma(\varepsilon)^-1$. However, $\varepsilon^*(\varphi) = |\tau(\varepsilon)z|^2 + \operatorname{Im}(\sigma(\varepsilon)u)^{-1}$, hence this $\varepsilon^*(\varphi)=\tau(\varepsilon)^2\varphi=\sigma(\varepsilon)^{-1}\varphi$. We have found an Kähler form on $\mathbb{C}\times\mathbb{H}$. ■

Curves on Inoue surface

THEOREM: The Inoue surface $M := (\mathbb{C} \times \mathbb{H})/\Gamma$ **has no complex curves**

Proof. Step 1: Consider on $\mathbb{C} \times \mathbb{H}$ a function $\varphi(z, z') := \log \operatorname{Im}(z')$. Since Γ multiplies $\operatorname{Im}(z')$ by a number, **the form $d\varphi$ is Γ-invariant.** Let θ be the corresponding 1-form on M.

Step 2: The 2-form $\omega_0 := d(I\theta)$ has Hodge type (1,1) and **is positive definite on the leaves of the foliation $\{z\} \times \mathbb{H} \subset \mathbb{C} \times \mathbb{H}$**. Indeed,

$$\omega_0 = \sqrt{-1}\,\partial\bar\partial \log \varphi = \sqrt{-1}\,\frac{dz' \wedge d\bar z'}{|\operatorname{im} z'|^2},$$

where ω_0 is the Poincare metric on $\mathbb{H}$.

Step 3: Let $\Sigma \subset TM$ be the null-space of the form ω_0. **It is a holomorphic, involutive foliation,** whose leaves are obtained from $\mathbb{C} \times \{z'\} \subset \mathbb{C} \times \mathbb{H}$

Step 4: For any complex curve C on M, $\int_C \omega_0 = 0$, because ω_0 is exact. Therefore, C is tangent to a leaf of Σ. **It remains to show that Σ has no compact leaves.**

Curves on Inoue surfaces

Step 3: Let $\Sigma \subset TM$ be the null-space of the form ω_0. **It is a holomorphic, involutive foliation,** whose leaves are obtained from $\mathbb{C} \times \{z'\} \subset \mathbb{C} \times \mathbb{H}$

Step 4: For any complex curve C on M, $\int_C \omega_0 = 0$, because ω_0 is exact. Therefore, C is tangent to a leaf of Σ. **It remains to show that Σ has no compact leaves.**

Step 5: Let Σ_0 be a leaf of Σ. Its preimage in $\mathbb{C} \times \mathbb{H}$ contains the set

$$\tilde{\Sigma}_0 := \bigcup_{z \in \mathbb{C}, \zeta \in \mathcal{O}_K^+} \Big(z, (z' + \sigma(\zeta)) \Big)$$

where $z' \in \mathbb{H}$ is a fixed point. Since the image of σ is dense in $\mathbb{R}$, **the closure $\tilde{\Sigma}_0$ contains $\mathbb{C} \times \mathbb{R} \times \operatorname{Im}(z')$.**

Step 6: Therefore, **the closure $\Sigma_0 \subset M$ is at least 3-dimensional,** hence **Σ has no compact leaves**.

Oeljeklaus-Toma manifolds

Let K be a number field which has $2t$ complex embedding denoted $\tau_i, \overline{\tau}_i$ and s real ones denoted σ_i, $s > 0$, $t > 0$.

Let $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \bigcap_i \sigma_i^{-1}(\mathbb{R}^{>0})$. Choose in $\mathcal{O}_K^{*,+} \cong \mathbb{Z}^{t+1-1}$ a free abelian subgroup $\mathcal{O}_K^{*,U} \cong \mathbb{Z}^s$ such that the quotient $\mathbb{R}^s/\mathcal{O}_K^{*,U}$ is compact, where $\mathcal{O}_K^{*,U}$ is mapped to $\mathbb{R}^t$ as $\xi \longrightarrow \Big(\log(\sigma_1(\xi)), ..., \log(\sigma_t(\xi))\Big)$. Let $\Gamma := \mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,U}$.

DEFINITION: An Oeljeklaus-Toma manifold is a quotient $\mathbb{C}^t \times \mathbb{H}^s/\Gamma$, where $\mathcal{O}_K^+$ acts on $\mathbb{C}^t \times \mathbb{H}^t$ as

$$\zeta(x_1, ..., x_t, y_1, ..., y_s) = \Big(x_1 + \tau_1(\zeta), ..., x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), ..., y_s + \sigma_s(\zeta)\Big),$$

and $\mathcal{O}_K^{*,U}$ as

$$\xi(x_1, ..., x_t, y_1, ..., y_s) = \Big(\tau_1(\xi)x_1, ..., \tau_t(\xi)x_t, \sigma_1(\xi)y_1, ..., \sigma_t(\xi)y_t\Big)$$

Oeljeklaus-Toma manifolds are LCK

THEOREM: (Oeljeklaus-Toma) The OT-manifold $M := \mathbb{C}^t \times \mathbb{H}^s/\Gamma$ **is a compact complex solvmanifold**. When $t = 1$, it is locally conformally Kähler. When $s = 1, t = 1$, it is an Inoue surface of class S^0.

Proof: We write the automorphic Kähler metric on $\mathbb{C} \times \mathbb{H}^s$ as $dd^c\varphi$, where $\varphi(x, \zeta_1, ..., \zeta_s) = |x|^2 + \prod_{i=1}^s \operatorname{Im}(\zeta_i)^{-1}$. The function φ is clearly plurisobharmonic (it is Poincare metric on each $\mathbb{H}$, and Euclidean on $\mathbb{C}$), and $dd^c\varphi$ is $\mathcal{O}_K^+$-invariant. Any $\xi \in \mathcal{O}_K^{*,+}$ multiplies $|x|^2$ by $A := \tau(\xi)^2$ and $\prod_{i=1}^s \operatorname{Im}(\zeta_i)^{-1}$ by B^{-1}, where $B := \prod_{i=1}^s \sigma_i(\xi)^{-1}$. However, $AB = N(\xi) = 1$, because the norm $N(\xi)$ is integer and invertible. ■

Complex geometry of Oeljeklaus-Toma manifolds

THEOREM: Let K be a number field which has s real embeddings and $2t$ complex ones, $t=1$, $s>0$. **Then the corresponding Oeljeklaus-Toma manifold has no non-trivial complex subvarieties**.

Proof. Step 1: Consider on $\mathbb{C}\times\mathbb{H}^t$ a function $\varphi(z,z_1,...,z_s) := \prod_i \operatorname{Im}(z_i)$. Since Γ multiplies $\operatorname{Im}(z_i)$ by a number, **the form $d\log\varphi$ is Γ-invariant.** Let θ denote the corresponding 1-form on $M=\mathbb{C}\times\mathbb{H}^s/\Gamma$.

Step 2: The 2-form $\omega_0 := d(I\theta) = dd^c\log\varphi$ has Hodge type (1,1) and **positive definite on the leaves of the foliation $\{z\}\times\mathbb{H}^t\subset\mathbb{C}\times\mathbb{H}^t$**

$$\omega_0 = \sqrt{-1}\,\partial\bar\partial\log\varphi = \sqrt{-1}\sum_i \frac{dz_i\wedge d\bar z_i}{|\operatorname{im} z_i|^2}.$$

Also, $\omega_0 \geqslant 0$.

Step 3: Let $\Sigma\subset TM$ be the null-foliation of ω_0 (the foliation generated by the null eigenspace). **It is a holomorphic, involutive, smooth 1-dimensional foliation,** with the leaves which are obtained from $\mathbb{C}\times\{(z_1,...,z_s)\}\subset\mathbb{C}\times\mathbb{H}^s$.

Step 4: For any complex k-dimensional subvariety $C \subset M$, the integral $\int_C \omega_0^k = 0$, because ω_0 is exact. Therefore, C is at each point tangent to a leaf of Σ. **Since Σ is 1-dimensional, this means that C contains at least one leaf of Σ.**

Step 5: It remains to show that any variety which contains a leaf of Σ coincides with M.

Step 6: Let Σ_0 be a leaf of Σ. Its preimage in $\mathbb{C} \times \mathbb{H}^s$ contains a set

$$\tilde{\Sigma}_0(z_1, ..., z_s) := \bigcup_{z \in \mathbb{C}, \zeta \in \mathcal{O}_K^+} \Big(z, (z_1 + \sigma_1(\zeta), ..., z_s + \sigma_s(\zeta))\Big)$$

where $z_1, ..., z_s \in \mathbb{H}^s$ is some fixed point.

Step 7: We reduced the theorem to the following statement

CLAIM: A closure of $\tilde{\Sigma}_0(z_1, ..., z_s)$ contains a set

$$Z_{\alpha_1, ..., \alpha_s} := \{(\zeta, \zeta_1, ..., \zeta_s) \ \mid \ \operatorname{im} \zeta_i = \alpha_i, i = 1, ..., s\}$$

where $\alpha_i = \operatorname{im} z_i$.

Indeed, **the smallest complex subspace containing $T_x Z_{\alpha_1, ..., \alpha_s}$ is $T_x M$.**

The adele ring

The previous claim is immediately implied by the following statement, applied to **the set $\rho_1, ..., \rho_m$ of all real embedings.**

Theorem 1 Let $K:\mathbb{Q}$ be a number field with has $2t$ complex embeddings $\tau_1, \overline{\tau}_1, ...$ and s real ones, $\sigma_1, ..., \sigma_t$, $\rho_1, ..., \rho_m$ – embeddings K to $\mathbb{C}$ or $\mathbb{R}$, and each of τ_i and σ_i appears once, except one. Consider the map $R: K \longrightarrow \mathbb{R}^a \times \mathbb{C}^b$, $R(\xi) := \rho_1(\xi), ..., \rho_m(\xi)$. **Then the image of $\mathcal{O}_K$ is dense in $\mathbb{R}^a \times \mathbb{C}^b$.**

The proof is based on the **strong approximation theorem** (which is a "modern version" of Chinese remainders theorem).

DEFINITION: Adelic group $\mathcal{A}_K$ is a subset of the product $\prod_\nu K_\nu$ of all completions of K for all equivalence classes ν of absolute value functions, consisting of sequences $(x_{\nu_1}, ..., x_{\nu_n}, ...)$ where $|x_{\nu_i}| \leqslant 1$ for all i except the finite number.

REMARK: Tikhonov's theorem **implies that $\mathcal{A}_K$ is locally compact.**

The strong approximation theorem

Strong approximation theorem: Consider the natural embedding $K \subset \mathcal{A}_K$. **Then its image is a discrete, cocompact subgroup.** Moreover, the projection of $\mathcal{A}_K \xrightarrow{P_{\nu_0}} \prod_{\nu\neq\nu_0} K_\nu$ to the product of all completions except one **maps K to a dense subset of $R_{\nu_0}(\mathcal{A}_K)$.**

REMARK: Further on, K is considered as a **subring of $\mathcal{A}_K$**.

Proof of Theorem 1. Step 1: Let $\mathcal{O}_{A_K}$ be a ring of all **integer adeles**, that is, such $(x_{\nu_1}, ..., x_{\nu_n}, ...) \in \mathcal{A}_K$, that $|x_{\nu_i}| \leqslant 1$ for each non-archimedean absolute value. **Then $\mathcal{O}_K = K \cap \mathcal{O}_{A_K}$.**

Step 2: Let now $P: \mathcal{A}_K \longrightarrow \mathcal{A}_1$ be a projection of $\mathcal{A}_K$ to the product of all completions except one archimedean. **Since $\mathcal{O}_{A_K}$ is open in $\mathcal{A}_K$, its projection to $\mathcal{A}_1$ is open in $\mathcal{A}_1$** (the projection **is an open map**).

Step 3: We obtain that the image $P(K)\cap P(\mathcal{O}_{A_K})$ is dense in $P(\mathcal{O}_{A_K})$. **From Step 1, we obtain that $P(K) \cap P(\mathcal{O}_{A_K})$ coinsides with $P(\mathcal{O}_K)$.**

Step 4: We proved that $P(\mathcal{O}_K)$ is dense in $\mathcal{A}_1 \cap P(\mathcal{O}_{A_K})$. **Therefore, its projection to the product of all archimedean completions except one is also dense.**