Locally conformally Kähler manifolds

lecture 15: complex surfaces

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Complex surfaces: Riemann-Roch theorem

DEFINITION: A complex surface is a compact, complex manifold of complex dimension 2. It is called **minimal** if it does not contain rational curves with self-intersection -1.

DEFINITION: Holomorphic Euler characteristic of a coherent sheaf F is $\sum_i (-1)^i \dim H^i(F)$.

THEOREM: (Riemann-Roch-Hirzebruch) Let L be a holomorphic line bundle on a complex surface X, and K_X its canonical bundle. **Then** $\chi(L) = \chi(\mathcal{O}_X) + \frac{(L-K_X,L)}{2}$**, and** $\chi(\mathcal{O}_X) = (c_1(X)^2 - c_2(X))/12$. ■

COROLLARY: Let L be a line bundle on a complex surface, $(L,L) = d > 0$. Then $\dim H^0(L^k)$ or $\dim H^0(L^{-k} \otimes K_X)$ grows at least quadratically with k.

Proof: $\chi(L^k) = dk^2 + ak + b$, where a, b are independent from k. However, $\chi(L^k) = h^0(L^k) - h^1(L^k) + h^2(L^k)$, hence either $h^0(L^k)$ or $h^2(L^k)$ grows quadratically. Finally, $h^2(L^k) = h^0(L^{-k} \otimes K_X)$ by Serre's duality. ■

Douady space

DEFINITION: Let M be a metric space, and $S, S' \subset M$ two subsets. The **Hausdorff distance** between S and S' is an infimum of all ε such that S lies in ε-neighbourhood of S' and S' lies in ε-neighbourhood of S.

DEFINITION: Given a complex subvariety $S \subset M$, **the Douady space** of deformations of S in M is the set of all complex subvarieties in the same cohomology class, equipped with topology induced by the Hausdorff metric $d_H(S, S')$.

CLAIM: Douady space is a complex analytic variety. ■

Base point set of a bundle

DEFINITION: Let L be a holomorphic line bundle. Define **base point set** $\operatorname{bps}(L)$ as the set of all $x \in M$ such that any section of L vanishes in x.

DEFINITION: A **movable divisor** is a divisor with positive-dimensional Douady set (that is, movable in a family).

CLAIM: Let L be a holomorphic line bundle, t its non-zero section, and D its zero divisor. **Then $D = D_0 \cup D_1$, where $D_0 \subset \operatorname{bps}(L)$, and D_1 is a union of movable divisors.**

Proof: Let $x \in D \backslash \operatorname{bps}(L)$. Then there exists a continuous family of divisors such that $D_t \not\ni x$, hence the component of D containing x is movable. ■

CLAIM: Let C, C' be movable divisors without common components on a surface. **Then $C \cap C'$ is a finite set.** ■

Finite correspondences

DEFINITION: Let $Z \subset M \times M'$ be an irreducible subvariety. Denote by π, π' the corresponding projections. It is called **a birationally finite correspondence** if $\pi^{-1}(m)$ and $\pi'^{-1}(m')$ is finite for general m, m'.

Proposition 1: Let L be a holomorphic line bundle on a surface, $c_1(L)^2 > 0$, $h^0(L) > 2$. Consider a subvariety $Z \subset M \times \mathbb{P}H^0(L)^*$,

$$Z = \{x \in M, t \in H^0(L)^* \ \ | \ \ V_x \subset \ker t\},$$

where $V_x := \{h \in H^0(L) \ \ | \ \ h|_x = 0\}$ is the space of all sections vanishing in x. **Then Z is a birationally finite correspondence** between M and $\pi_2(Z)$, where $\pi_2 : Z \longrightarrow \mathbb{P}H^0(L)^*$ is a projection.

Proof. Step 1: Let $t \in H^0(L)\backslash 0$, and let $Z_t \subset M$ be the zero divisor of t, and $H_t \subset \mathbb{P}H^0(L)^*$ the dual hypersurface. Then $\pi_1(\pi_2^{-1}(H_t)) = Z_t$. For any $t_1 \neq t_2 \in H^0(L)\backslash 0$, denote by W_{t_1,t_2} the intersection $H_{t_1} \cap H_{t_2}$. Then $\pi_1(\pi_2^{-1}(W_{t_1,t_2})) = Z_{t_1} \cap Z_{t_2}$.

Step 2: The intersection $Z_{t_1} \cap Z_{t_2}$ is a union of base point divisors and intersection of movable divisors. Since it is 0-dimensional outside of $\operatorname{bps}(L)$, the correspondence is finite outside of $\operatorname{bps}(L)$. ■

Finite correspondences (2)

THEOREM: Let M be a complex surface, admitting a birationally finite correspondence to M'. Then **M' is Kähler (projective) if and only if M is Kähler (projective).**

The proof is based on currents and Hahn-Banach separation theorem.

REMARK: This is true **only for surfaces**!

Currents

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M} \left(|f| + |\nabla f| + ... + |\nabla^k f|\right)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

REMARK: Currents are forms with coefficients in generalized functions.

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact). Montel spaces are **reflexive** (the map to its double dual with strong topology is an isomorphism).

CLAIM: De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions

REMARK: In the literature, this is sometimes called $(n-p,n-q)$-currents.

CLAIM: The Dolbeault lemma holds on (p,q)-currents, and **the $\overline{\partial}$-cohomology are the same as for forms.**

Positive forms and currents

DEFINITION: A **weakly positive (p,p)-form** is a real (p,p)-form η which satisfies $\eta(x_1, Ix_1, x_2, Ix_2, ...x_p, Ix_p) \geqslant 0$ for all $x_1, ...n_p \in TM$. **The set of weakly positive (p,p)-forms is a convex cone.**

DEFINITION: A **weakly positive (p,p)-current** is a current taking non-negative values in weakly positive compactly supported $(n-p, n-p)$-forms.

DEFINITION: A **positive generalized function** is a generalized function taking non-negative values on all positive volume forms.

REMARK: Positive generalized functions are C^0-continuous. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and all measures are obtained this way.

COROLLARY: A weakly positive (p,p)-current is C^0-continuous.

Closed positive currents and psh functions

DEFINITION: Let $Z \subset M$ be a complex analytic subvariety. **The current of integration** $[Z]$ is the current $\alpha \longrightarrow \int_Z \alpha$. **It is closed and positive** (Lelong).

REMARK: (Poincare-Lelong formula) $\frac{\sqrt{-1}}{\pi} dd^c \log |\varphi| = [Z_\varphi]$, where Z_φ is a divisor of a holomorphic function φ.

DEFINITION: A locally integrable function $f: M \longrightarrow [\infty, \infty[$ is called **plurisubharmonic** (psh) if $dd^c f$ is a positive current.

CLAIM: (a local dd^c-lemma) **Locally, every positive, closed (1,1)-current is obtained as $dd^c f$,** for some psh function f.

EXERCISE: Prove that **a locally integrable plurisubharmonic function on a compact complex manifold is constant.**

Hahn-Banach separation theorem and its applications

THEOREM: (Hahn-Banach separation theorem)
Let V be a locally convex topological vector space, $A \subset V$ an open convex subset, and $W \subset V$ a closed subspace. Assume that $W \cap A = \emptyset$. Then there exists a continuous functional $\xi \in V^*$ such that $\xi(W) = 0$ and $\xi(A) > 0$. ■

THEOREM: (Harvey-Lawson)
Let M be a compact non-Kähler complex manifold. **Then M admits an exact $2n-2$-current such that its $(n-1, n-1)$-part is positive.**

REMARK: Converse is also true: if M admits such a current, M is non-Kähler **(prove this).**

Proof of Harvey-Lawson theorem. Step 1:
Let $A \subset \Lambda^{1,1}M$ be the set of all strictly positive forms, and W the space of all closed (1,1)-forms. **Hahn-Banach separation theorem produces a current $\xi^{1,1} \in D^{n-1,n-1}(M)$ such that $\xi^{1,1}(A)$ is positive and $\xi^{1,1}(W) = 0$.** Clearly, $\xi^{1,1}(A) > 0 \Leftrightarrow \xi^{1,1}$ is positive.

Hahn-Banach separation theorem and its applications (2)

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Step 2: Consider the space $W_1 \subset \Lambda^2(M)$ generated by all closed furms and all (1,1)-forms. Extend $\xi^{1,1}$ to W_1 by taking $\xi^{1,1}(v) = 0$ for all closed v. Since $\xi^{1,1}$ vanishes on closed (1,1)-forms, it is well defined on W_1, and can be extended to a continuous functional on Λ^2 (Hahn-Banach extension theorem). We obtain **a $(2n-2)$-current ξ vanishing on closed forms and with positive (1,1)-part.**

Step 3: It remains to prove that ξ is exact. Since $\langle \xi, d\alpha \rangle = \pm \langle d\xi, \alpha \rangle = 0$ for all α, the current ξ is closed. However, a pairing of ξ with any closed form vanishes, hence **ξ is exact by Poincare duality.** ■

Gauduchon metrics and Hahn-Banach theorem

DEFINITION: A Hermitian metric ω on complex n-manifold is called **Gauduchon** if $dd^c\omega^{n-1} = 0$.

THEOREM: **Any compact complex manifold admits a Gauduchon metric.**

Step 1: Any strictly positive $(n-1, n-1)$-form is $(n-1)$-th power of a Hermitian form. Therefore, to construct a Gauduchon metric, it suffices to find a dd^c-closed strictly positive $(n-1, n-1)$-form.

Step 2: Let $A \subset \Lambda^{n-1,n-1}(M)$ be the cone of strictly positive $(n-1, n-1)$-forms, and $W := \ker dd^c$. If these sets don't intersect, we can find $\xi \in \Lambda^{1,1}(M)$ which is positive and satisfies $\langle \xi, \ker dd^c \rangle = 0$.

Gauduchon metrics and Hahn-Banach theorem (2)

THEOREM: Any compact complex manifold admits a Gauduchon metric.

Step 2: Let $A \subset \Lambda^{n-1,n-1}(M)$ be the cone of strictly positive $(n-1,n-1)$-forms, and $W := \ker dd^c$. If these sets don't intersect, we can find $\xi \in \Lambda^{1,1}(M)$ which is positive and satisfies $\langle \xi, \ker dd^c \rangle = 0$.

Step 3: Since $\operatorname{im} d \subset \ker \ker dd^c$, this gives $\langle \xi, \operatorname{im} d \rangle = 0$, hence ξ is closed. Define **Aeppli cohomology** as

$$H^{p,q}_{AE}(M) := \frac{\ker dd^c\Big|_{\Lambda^{p,q}(M)}}{\operatorname{im} d + \operatorname{im} d^c}.$$

The Poincare pairing $H^{p,q}_{AE}(M) \times H^{n-p,n-q}_{BC}(M) \longrightarrow \mathbb{C}$, where $H^{*,*}_{BC}(M)$ is Bott-Chern cohomology, is non-degenerate **(prove this)!** Since ξ is orthogonal to $H^{p,q}_{AE}(M)$ under this pairing, its class in $H^{1,1}_{BC}(M)$ vanishes, hence $\xi = dd^c f \geqslant 0$.

Step 4: There are no globally plurisubharmonic generalized functions on a compact manifold **(prove it!)**. ■

REMARK: In fact, **there exists a Gauduchon metric in each conformal class, and it is unique up to a constant.** This is proven using elliptic equations and E. Hopf's strict maximum principle.

Finite correspondences and Kählerness

THEOREM: Let M_1 be a complex surface, admitting a birationally finite correspondence to M_2. Then **M_1 is Kähler if and only if M_2 is Kähler.**

Proof. Step 1: Let $\pi_1, \pi_2 : Z \longrightarrow M_i$ be the projection maps, ω_2 be a Kähler form on M_2, and $\zeta := \pi_{1*}\pi_2^*\omega$. Then ζ is a positive, closed (1,1)-current which is smooth and strictly positive at each point $z \in M_1$ where the correspondence Z is finite.

Step 2: Since ζ is infinite only around the points $z \in M_1$ where $\pi_1^{-1}(z)$ is positive-dimensional, ζ is smooth outside of a closed, finite set $\operatorname{sing}(\zeta)$.

Step 3: Let S_ε be an epsilon-neighbourhood of $\operatorname{sing}(\zeta)$. Using the local dd^c-lemma in S_ε, we could write $\zeta = dd^c f$. Since a maximum of plurisubharmonic functions is plurisubharmonic, we can replace ζ by a current which is equal to ζ outside S_ε and equal to $dd^c \max(f, -C)$ in S_ε, where $-C < f\big|_{\partial S_\varepsilon}$. The new ζ is positive, closed, and equal to $dd^c(\varphi)$ on S_ε.

Finite correspondences and Kählerness (2)

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Step 4: Consider a smooth convex function $\max_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}$ equal to $\max(x,y)$ for $|x-y| > \varepsilon$ and monotonous in each argument. Using $\max_\varepsilon$ instead of max in Step 3, we may assume that ζ is actually smooth.

Step 5: Suppose that M_1 is non-Kähler. Then Harvey-Lawson bring an exact current $\xi = d\alpha$ with positive (1,1)-part. The current $\zeta \wedge \xi$ is positive; on the other hand, it is exact, giving $0 = \int_M \zeta \wedge \xi > 0$ – a contradiction. ■

REMARK: In dimension > 2, this theorem is false; it should be clear where the argument fails.

Kähler cone and Nakai-Moishezon-Demailly-Paun theorem

THEOREM: (Nakai-Moishezon-Demailly-Paun)
Let M be a Kähler surface, and

$$K := \{\alpha \in H^{1,1}(M) \ \mid \ \alpha^2 > 0, \ \ \alpha\Big|_{\text{all complex curves}} > 0\}.$$

Then K is a Kähler cone of M unless M has no complex curves. In the latter case, Kähler cone of M is one of two connected components of K. ■

THEOREM: Let M_1 be a complex surface, admitting a birationally finite correspondence to M_2. Then **M_1 is projective if and only if M_2 is projective.**

Proof. Step 1: Using the Harvey-Lawson argument such as above, we prove that **M_1 is Kähler.** Let ω be a rational Kähler form on M_2, and $\zeta := \pi_{1*}\pi_2^*\omega$. **Then $\zeta^2 > 0$** by the same argument (with removal of isolated singularities) as used above.

Step 2: For any curve $C \subset M_1$, let C_Z be its proper preimage in Z. Then $\langle \xi, C\rangle = \int_{C_Z} \pi_2^*\omega > 0$. Then by Nakai-Moishezon-Demailly-Paun, **$[\xi]$ is a Kähler class.**

Step 3: This class is rational by construction, hence **M_1 is projective by Kodaira.** ■

Non-projective surfaces and the intersection form

THEOREM: Let M be a complex surface, and $NS(M)$ the group of all integer cohomology classes represented by closed $(1,1)$-currents. **Then M is non-projective if and only if the intersection form on $NS(M)$ is non-positive.**

Proof. Step 1: Suppose that $[\xi] \in NS(M)$ is an integer class with $[\xi]^2 = 0$, and L the line bundle with its curvature equal to ξ (it exists by dd^c-lemma). Then either $h^0(L^k)$ or $h^0(L^k \otimes K_M)$ grows quadratically with k by Riemann-Roch.

Step 2: Proposition 1 implies that M admits a birationally finite correspondence with a projective manifold, hence M is projective as shown above.

Step 3: Conversely, if M is projective, it admits a curve with positive self-intersection, namely, the hyperplane section. ■

DEFINITION: Elliptic surface is a complex surface M equipped with a holomorphic map $M \longrightarrow S$, with generic fiber a curve of genus 1.

Curves on non-projective surfaces

THEOREM: Let M be a non-projective surface. Then **either all curves on M are isolated, or M admits an elliptic fibration $\pi : M \longrightarrow S$,** with all curves belonging to the fibers of π.

Step 1: Since the intersection form on $NS(M)$ is non-positive, for any two distinct irreducible curves C, C' in the same cohomology class, the intersection $C \cap C'$ is empty. Therefore, all non-isolated curves C satisfy $C \cdot C = 0$.

Step 2: Given a family C_t of non-intersecting curves parametrized by $t \in D$, consider the corresponding fibration from a subset $M_0 := \bigcup_{t \in D} C_t \subset M$ to D. Since $NC_t = \pi^* TD\Big|_{C_t}$, the normal bundle to each C_t is trivial.

Step 3: Adjunction formula gives $TC_t \otimes NC_t = \Lambda^2 TM\Big|_{C_t}$, hence $\Omega^1 C_t = K_M\Big|_{C_t}$, where K_M is a canonical bundle. If $(K_M, C_t) = l \neq 0$, one would have $(K_M + tC_t, K_M + tC_t) = (K_M, K_M) + 2tl > 0$ for appropriate choice of t. Then M would be projective. Therefore, $(K_M, C_t) = 0 = \deg \Omega^1 C_t$. This implies that all smooth fibers of π are elliptic curves.

Step 5: Existence of elliptic fibration $M \longrightarrow B$ would follow if we show that the deformation space of C_t is compact for any curve on a complex surface (see the next slide).

Gromov's compactness theorem

Step 6: The same argument as in Step 4 is used to show that $(C_t, v) = 0$ for any $v \in NS(M)$, hence any irreducible curve belongs to a fiber of π. ■

THEOREM: (Gromov's compactness theorem)
Let (M, I, ω) be a compact (almost) complex Hermitian manifold, $\mathfrak{D}$ the space of all (pseudo-) holomorphic curves on M, with topology induced by the Hausdorff metric, $p > 0$ a real number, and $\mathfrak{D}_p \subset \mathfrak{D}$ the space of all curves S with $\operatorname{Vol}(S) := \int_S \omega \leqslant p$. Then $\mathfrak{D}_p$ is compact.

Moduli of curves on complex surfaces

THEOREM: Let M be a compact complex surface, $C \subset M$ a curve, and B a connected component of its Douady space. **Then B is compact.**

Proof. Step 1: Fix a Hermitian metric ω on M. By Gromov's theorem, the space of curves of bounded volume is compact. **It remains to show that the volume stays bounded on each connected component of the Douady space,** for appropriate choice of ω. Notice that this is vacuously true when ω is Kähler, because then the volume is a cohomological invariant.

Step 2: Consider the incidence variety $Z \subset M \times B$ consisting of pairs $C \in B, x \in C \subset M$, and let $\pi_1 : Z \longrightarrow M$, $\pi_2 : Z \longrightarrow B$ be the standard projections. Denote by $\mathrm{Vol} : B \longrightarrow \mathbb{R}^{>0}$ the volume function, $\mathrm{Vol}(C) = \int_C \omega$. Then $\mathrm{Vol} = \pi_{2*}\pi_1^*\omega$.

Step 3: Choose now ω Gauduchon. Then $dd^c \mathrm{Vol} = \pi_2 * \pi_1^* dd^c\omega = 0$. Therefore, Vol has no local minima or local maxima on B. However, Vol is a proper function by Gromov's theorem, hence it has to reach minimum somewhere. **Therefore,** $\mathrm{Vol} = const$**.** Now, B is compact by Gromov's theorem. ■

Class VII surfaces (a survey)

DEFINITION: Define the **Kodaira dimension** of a manifold M as $\kappa(M) := \limsup_n \frac{\log h^0(K_M^n)}{n}$.

DEFINITION: A **Kodaira class VII surface** is a surface with $\kappa(M) = -\infty$ and $b_1(M) = 1$.

EXAMPLE: All Hopf surfaces and all Inoue surfaces are Kodaira class VII.

DEFINITION: **Kato surface** is a surface M which contains a 3-dimensional sphere S^3 such that $M\backslash S^3$ is connected, and a neighbourhood of S^3 is biholomorphic to a neighbourhood of standard S^3 in $\mathbb{C}^2$ ("global spherical shell").

THEOREM: All Kato surfaces are class VII.

The rest of classification theorems (a survey)

THEOREM: Let M be a non-projective Kähler minimal surface. Then M is elliptic, or isomorphic to a K3 or a torus.

THEOREM: Let M be a non-Kähler elliptic minimal surface. Then M is isotrivial (all fibers are isomorphic) and Vaisman.

THEOREM: Let M be a non-Kähler non-elliptic surface. Then M is class VII.

THEOREM: (Bogomolov) All class VII surfaces with $b_2 = 0$ are Inoue or Hopf.

THEOREM: (Andrei Teleman) All class VII surfaces with $b_2 = 1$ are Kato.

CONJECTURE: (GSS conjecture) All minimal class VII surfaces with $b_2 > 0$ are Kato.

THEOREM: (Brunella) All Kato surfaces are LCK.