LCK manifolds 1: local systems and Morse-Novikov cohomology

Exercise 1.1. Let $\lambda > 0$ be a real number. Define **weight λ homogeneous forms** on $\mathbb{R}^n \backslash 0$ as differential forms η which satisfy $\rho_t^* \eta = \lambda^t \eta$, where ρ_t is a homothety map $z \longrightarrow tz$, $t > 0$. Prove that a closed weight λ form is exact for any $\lambda \neq 1$.

Exercise 1.2. Let $M = \mathbb{R}^n \backslash 0 / (x \sim 2x)$ be a Hopf manifold, θ a closed, non-exact 1-form, $d_\theta = d - \theta$, and $H^*_\theta(M)$ cohomology of the complex $\Lambda^*(M), d_\theta$ ("Morse-Novikov cohomology"). Prove that $H^i_\theta(M) = 0$ for all i.

Hint. Use the previous exercise.

Exercise 1.3. Let $M = X \times S^1$, $\pi : \ M \longrightarrow S^1$ the standard projection, $\theta := \pi^* dt$. Prove that $H^i_\theta(M) = 0$ for all i.

Exercise 1.4. Define **projectively flat** connection on a vector bundle B over M as a connection with curvature $\Theta \in \Lambda^2 M \otimes \operatorname{End}(B)$ which satisfies $\Theta \in \Lambda^2 M \otimes \mathsf{Id}_B$. Prove that projectively flat connections on a bundle of rank r are equivalent to representations $\pi_1(M) \longrightarrow PGL(n)$.

Exercise 1.5. Let M be a compact manifold, and θ a closed, non-exact, nowhere vanishing one-form. Prove that $H^*_{\lambda\theta}(M) = 0$ for all $\lambda \in \mathbb{R}$, except a finite number.

Exercise 1.6. (Moser stability for LCS)
Let (M, ω, θ) be a compact locally conformally symplectic manifold, ω_t a continuous deformation of ω satisfying $d\omega_t = \omega_t \wedge \theta$ and $[\omega_t] = \mathsf{const}$, where $[\omega_t] \in H^2_\theta(M)$ the Morse-Novikov class of ω. Find a flow of diffeomorphisms mapping ω to ω_t.

Hint. Use the usual Moser stability argument. Interpret forms with coefficients in the weight bundle as equivariant forms on the universal cover.