LCK manifolds 3: Homotheties on Riemannian manifolds

Exercise 3.1. Let ∇ be a torsion-free connection on a manifold M, and X a vector field which satisfies $\nabla(X) = \mathsf{Id}_{TM}$.

a. Prove that either the holonomy group of ∇ is non-compact, or M is non-compact.

b. Find an example of such connection when M is compact.

Exercise 3.2. Let ∇ be a Levi-Civita connection on M, $R \in \Lambda^2 M \otimes \operatorname{End}(TM)$ its curvature tensor, and and $X \in TM$ a vector field which satisfies $\nabla(X) = \mathsf{Id}_{TM}$. Prove that $R(X,Y) = 0$ for any $Y \in TM$.

Hint. Compute $\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X$.

Exercise 3.3. Let ∇ be the Levi-Civita connection on a Riemannian manifold (M,g) and X a vector field satisfying $\nabla(X) = \mathsf{Id}_{TM}$.

a. Prove that (M,g) is locally isometric to a Riemannian cone.

b. Let $R \in \Lambda^2 M \otimes \operatorname{End}(TM)$ be the curvature tensor. Prove that $\operatorname{Lie}_X(R) = 0$ and $\nabla_X(R) = -2R$.

c. Suppose that (M,g) is Einstein: $\operatorname{Ric}(M) = cg$. Prove that $c = 0$.

Exercise 3.4. Let X be a complete Riemannian manifold, $M = C(X)$ its Riemannian cone, and $\phi: \ M \longrightarrow M$ an isometry. Prove that ϕ is induced by an isometry of X.