LCK manifolds 4: Sasakian manifolds

Exercise 4.1. Let S be a regular Sasakian manifold, $X = S/\operatorname{\mathsf{Reeb}}$, $\pi_1(X) = 0$, $b_2(X) = 1$. Prove that $\pi_1(S)$ is a finite cyclic group.

Exercise 4.2. Let G be a finite cyclic group acting on a projective manifold X. Find a Sasakian manifold S such that $S/\operatorname{\mathsf{Reeb}} = X/G$.

Exercise 4.3. Let S be a regular Sasakian manifold, $S/\operatorname{\mathsf{Reeb}} = T^2$ (compact complex torus of dimension 1). Prove that $\pi_1(S)$ is non-abelian.

Exercise 4.4. Let S be a regular Sasakian manifold, and $X := S/\operatorname{\mathsf{Reeb}}$. Assume that $\pi_1(X) = \mathbb{Z}^2$, $\pi_2(X) = 0$. Prove that $\pi_1(S)$ is non-abelian.

Exercise 4.5. Let S be a Sasakian manifold, H a group of Sasakian automorphisms, and $H_{\mathbb{C}}$ the corresponding complex Lie group acting on $C(S)$ by holomorphic automorphisms. Prove that $\dim_{\mathbb{R}} H = \dim_{\mathbb{C}} H_{\mathbb{C}}$.

Definition 4.1. Let X be a projective manifold, L a positive line bundle, $\lambda > 0$, and $\operatorname{Tot}(L^*)$ the space of all non-zero vectors in L. The quotient $\operatorname{Tot}(L^*)/x \sim \lambda x$ is called **a regular Vaisman manifold**; it is equipped with a natural LCK structure (see Lecture 4).

Exercise 4.6. Let (M, ω, θ) be a compact LCK manifold. Since $d_\theta(\omega) = 0$, ω represents a cohomology class $[\omega] \in H^2_\theta(M)$, called **the Morse-Novikov class of** M. Prove that $[\omega] = 0$ when (M, ω, θ) is a regular Vaisman manifold.