LCK manifolds 5: Groups of automorphisms

Exercise 5.1. Let M be a compact Riemannian manifold.

a. Show that the group of isometries of M is compact.

b. Prove that it is a Lie group, and its Lie algebra is an algebra of Killing vector fields.

Exercise 5.2. Let M be a compact complex manifold, and G its group of holomorphic automorphisms, equipped with a natural (compact-open) topology. Prove that G is a Lie group, and its Lie algebra is an algebra of holomorphic vector fields.

Exercise 5.3. Let (M, I, ω) be a compact Kähler manifold, and $\mathfrak{g} \subset TM$ a subalgebra consisting of all vector fields $X \in TM$ such that $\operatorname{Lie}_X I = \operatorname{Lie}_X \omega = 0$ and $\operatorname{Lie}_{IX} I = \operatorname{Lie}_{IX} \omega = 0$.

a. Prove that it is a subalgebra.

b. Prove that for all $X \in \mathfrak{g}$, one has $\nabla(X) = 0$.

c. Prove that $\mathfrak{g}$ is Abelian.

Definition 5.1. Let M be a compact Kähler manifold. Define an equivalence relation $x \sim y$ on M as follows: two points x, y are equivalent if for any path γ connecting x to y and any harmonic form α representing an integer cohomology class, the number $\int_\gamma \alpha$ is integer. The quotient $M/\sim$ is called **the Albanese variety** of M, denoted by $\operatorname{Alb}(M)$.

Exercise 5.4. Prove that $\operatorname{Alb}(M)$ is a compact complex torus, and the projection $M \longrightarrow \operatorname{Alb}(M)$ is a holomorphic map.

Exercise 5.5. Let M be a compact Kähler manifold, $\operatorname{Aut}(M)$ the group of its holomorphic automorphisms, $\mathfrak{g} \subset TM$ the Lie algebra of vector fields satisfying $\nabla(X) = 0$, and G is Lie group.

a. Prove that $G \subset \operatorname{Aut}(M)$.

b. Prove that G is a normal subgroup in $\operatorname{Aut}(M)$.

c. Prove that all its orbits are compact complex tori in M.

d. Let $G_1 \subset \operatorname{Aut}(M)$ be the group of automorphisms acting trivially on $\operatorname{Alb}(M)$. Prove that $G_1 \cap G = 0$.

e. Prove that $\operatorname{Aut}(M)$ is a semidirect product of G and G_1.