LCK manifolds 6: Orbifolds

Definition 6.1. **Orbipoints** are points of an orbispace with $\operatorname{Mor}(x,x)$ non-trivial. **An order** of an orbipoint is $|\operatorname{Mor}(x,x)|$. The group $\operatorname{Mor}(x,x)$ is called **the monodromy group of an orbipoint**.

Exercise 6.1. Prove that a 1-dimensional complex orbifold is uniquely defined by the following data: a smooth complex curve M, some orbipoints x_i marked on M, and order $p_i \in \mathbb{Z}^{>1}$ of monodromy at each x_i.

Exercise 6.2. Let S be a quasiregular Sasakian manifold, and $X := S/\mathsf{Reeb}$ the corresponding orbifold. Prove that monodromy group of each orbipoint is a cyclic group.

Exercise 6.3. Let $\tilde M:\ \mathbb{C}^2\backslash 0$, and $\mathbb{C}^*$ act on $\tilde M$ as $h_t(x,y) = (tx, t^2y)$. Find a Vaisman metric on $\tilde M/h_\lambda$, where $\lambda > 1$ is a fixed number. Prove that $\tilde M/\mathbb{C}^*$ is $\mathbb{C}P^1$ with one orbipoint of order 2.

Exercise 6.4. Let $X = \mathbb{C}P^1$ with two orbipoints of order p and q. Find a quasiregular Sasakian manifold S with $S/\mathsf{Reeb} = X$.

Exercise 6.5. Define a covering in the orbifold category, and prove existence of a universal covering.

Definition 6.2. An **orbifold fundamental group** $\pi_1^{\mathsf{orb}}(M)$ is a group of automorphisms of the universal covering compatible with the projection to M. The **topological fundamental group** $\pi_1^{\mathsf{top}}(M)$ is the fundamental group of M as of the topological space.

Exercise 6.6. Construct a monomorphism $\pi_1^{\mathsf{top}}(M) \longrightarrow \pi_1^{\mathsf{orb}}(M)$. Find an orbifold M such that $\pi_1^{\mathsf{top}}(M)$ is trivial, and $\pi_1^{\mathsf{orb}}(M)$ is non-trivial.

Exercise 6.7. Let M be a 1-dimensional complex orbifold with a complete Hermitian metric of constant negative curvature, and $\pi_1^{\mathsf{orb}}(M) = 0$. Prove that M is equivalent to a Poincare disk Δ and has no orbipoints.

Exercise 6.8. Let M be a complex curve of genus 1 with at least one orbipoint. Prove that the universal covering of M (in the orbifold category) is the Poincare disc Δ.

Exercise 6.9. Let M be $\mathbb{C}P^1$ with two orbipoints of order p and q. Find $\pi_1^{\mathsf{orb}}(M)$.