LCK manifolds 7: Hopf manifolds

Definition 7.1. Contraction of a manifold M to a point $x \subset M$ is an automorphism ϕ such that for any open subset $U \ni x$ with compact closure there exists $N > 0$ such that for all $n > N$, the map ϕ^n maps U to a compact subset $K \subset U$.

Exercise 7.1. Let $\phi : M \longrightarrow M$ be a contraction to x. Prove that $\lim_n \phi^n(m) = x$, for each $m \in M$.

Exercise 7.2. Let $\phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial map preserving 0, and with $D\phi\big|_0$ invertible with all eigenvalues $|\alpha_i| < 1$.

a. Prove that ϕ defines a contraction on an open ball $B_\varepsilon(0)$.

b. Consider the equivalence relation on $B_\varepsilon(0)\backslash 0$ generated by $x \sim \phi(x)$. Prove that $(B_\varepsilon(0)\backslash 0)/\sim$ is a complex manifold diffeomorphic to $S^1 \times S^{2n-1}$.

Definition 7.2. The manifold $B_\varepsilon(0)/\sim$ defined above is called **Hopf manifold**. When ϕ is linear, it is called **a linear Hopf manifold**. In dimension 2, Hopf manifold is also called **Hopf surface**, or **primary Hopf surface**. A **secondary Hopf surface** is a quotient of a Hopf surface by a finite group freely acting on it.

Exercise 7.3. Find secondary Hopf surfaces with $\pi_1(H) = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, for each $n \in \mathbb{Z}^{>0}$.

Exercise 7.4. Find a secondary Hopf surface with $\pi_1(H) = \mathbb{Z} \oplus S_3$, where S_3 is a symmetric group.

Exercise 7.5. Let H_1, H_2 be Hopf manifolds associated with holomorphic contractions P_1, P_2 of an open ball $B \subset \mathbb{C}^n$. Assume that H_1 is biholomorphic to H_2. Prove that there exists a biholomorphism $F : B \longrightarrow B$ giving $P_1 = FP_2F^{-1}$.

Definition 7.3. Let $A \in GL(n, \mathbb{C})$ be a linear map, written in a basis $x_i \in \mathbb{C}^n$ diagonally as $A(x_i) = \alpha_i x_i$. A vector $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{N}^n$, $|\lambda| > 1$ is called **resonant** for X if $\alpha_s = \prod_{i=1}^n \alpha_i^{\lambda_i}$.

Consider a polynomial map $P = (P^1, ..., P^n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, with each $P^i \in \mathbb{C}[x_1, ..., x_n]$; assume that the linear part of P is equal to A. We write each P^s as a sum of monomials $P^s_{h_1 h_2 ... h_n} = \alpha_{h_1 h_2 ... h_n} x_1^{h_1} x_2^{h_2} ... x_n^{h_n}$. A monomial $P^s_{h_1 h_2 ... h_n}$ is **resonant** if $(h_1, h_2, ..., h_n)$ is a resonant vector.

Exercise 7.6. Let $P = (P^1, ..., P^n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial map, $A \in GL(n, \mathbb{C})$ its linear part, and $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ a polynomial automorphism preserving 0 and satisfying $dF\big|_0 = \mathsf{Id}$. Assume that all monomials in $P - A$ are of degree $\geqslant d$, and let P_d, F_d be the degree d term of P, F.

a. Consider the polynomial map FPF^{-1}. Show that the degree d term of FPF^{-1} is equal to $P_d + AF_d - F_d A$.

b. Given P as above, prove that there following are equivalent: (i) there exists F such that the degree d terms of FPF^{-1} vanish (ii) all resonant monomials of degree d in P vanish.

Exercise 7.7. Let $P : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a polynomial mapping (x, y) to $(\frac{1}{2}x, \frac{1}{4}y + x^2)$, and H the corresponding Hopf surface. Prove that H is not biholomorphic to a linear Hopf surface.