LCK manifolds 8: Stein manifolds and normal families

Exercise 8.1. Let X, Y – complex manifold, and $M(X, Y)$ the space of all holomorphic maps from X to Y, with open-compact topology.

a. Prove that $M(\mathbb{C}P^1, \mathbb{C}P^1)$ is compact, or find a counterexample.

b. Let X be a complex manifold, and X a complex curve of genus > 1. Prove that $M(X, Y)$ is compact.

c. Let $X = Y = T^2$ be an elliptic curve. Prove that $M(X, Y)$ is compact or find a counterexample.

Definition 8.1. Let V be a topological vector space. A subset $K \subset V$ is called **bounded** if for any open neighbourhood $U \ni 0$, there exists $a \in \mathbb{R}$ such that $aU \supset K$. A topological vector space V is called **Montel** if any closed bounded subset $K \subset V$ is compact.

Exercise 8.2. Let M be a complex manifold, B a vector bundle, and $V = H^0(B)$ the space of holomorphic sections, with topology of uniform convergence on compacts.

a. Prove that V is a Montel space.

b. Prove that any metrizable Montel space is finite-dimensional.

c. Show that $H^0(B)$ is always finite-dimensional if M is compact.

Definition 8.2. A complex variety M is called **holomorphically convex** if for any infinite discrete subset $S \subset M$, there exists a holomorphic function $f \in \mathcal{O}_M$ which is unbounded on S. A holomorphically convex variety without complex subvarieties is called **Stein**.

Exercise 8.3. Let $U, V \subset \mathbb{C}^n$ be open, holomorphically convex subsets. Prove that $U \cap V$ is also holomorphically convex.

Exercise 8.4. Consider the **Reinhardt triangle**:

$$D := \{(z_1, z_2) \in \mathbb{C}^2 \quad | \quad 0 < |z_1| < |z_2| < 1\}.$$

Prove that D is holomorphically convex.

Exercise 8.5. Prove that any Stein manifold admits a proper holomorphic map to $\mathbb{C}^n$.