LCK manifolds 10: Levi form

From now until the end, $n > 1$, and all hypersurfaces are equipped with orientation.

Exercise 10.1. Consider **the Penrose hypersurface** $S := \{z \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 = 1 + |z_3|^2\}$. Prove that its Levi form is non-degenerate and find its signature.

Definition 10.1. A real-valued function ϕ on a complex manifold is called **pluriharmonic** if $dd^c\phi = 0$

Exercise 10.2. Let ϕ be a pluriharmonic function, c its regular value, and $S := \phi^{-1}(c)$. Prove that the Levi form Φ of S is vanishing (such S is called **Levi-flat**).

Exercise 10.3. Let ϕ be a non-zero real function on a complex manifold such that for some $a, b \in \mathbb{R}$, one has $a\phi \cdot dd^c\phi + b \cdot d\phi \wedge d^c\phi = 0$. Prove that ϕ^λ is pluriharmonic, for some non-zero $\lambda \in \mathbb{R}$.

Exercise 10.4. Let $M \subset \mathbb{C}^n$ be a holomorphically convex subset with smooth boundary S. Prove that the Levi form of S is semi-positive.

Exercise 10.5. Let $S \subset \mathbb{C}^n$ be a compact smooth real hypersurface, such that the Levi form Φ of S is non-degenerate. Prove that Φ is sign-definite.

Exercise 10.6. Let $S \subset \mathbb{C}^n$ be a smooth Levi-flat hypersurface. Prove that S is non-compact.

Exercise 10.7. Let M be a complex manifold, and $\phi: M \longrightarrow \mathbb{R}$ a smooth function satisfying $d\phi \wedge d^c\phi \wedge dd^c\phi = 0$. Prove that for any regular value c of ϕ, the preimage $\phi^{-1}(c)$ is Levi-flat.

Exercise 10.8. Let $M = C(S)/\langle\gamma\rangle$ be a Vaisman manifold, with $\gamma(s,t) = (\phi(s), \lambda t)$, where ϕ is a Sasakian automorphism.

a. Prove that there exists a holomorphic vector field $\vec{r}$ such that $e^{\vec{r}} = \gamma$ if and only if ϕ lies in the connected component of the Lie group of Sasakian automorphisms.

b. Find a Vaisman manifold $M = C(S)/\langle\gamma\rangle$ such that $\gamma \neq e^{\vec{r}}$ for any holomorphic $\vec{r}$.