LCK manifolds 11: CR vs. Sasakian

Definition 11.1. Let M be a compact Kähler manifold. Define an equivalence relation $x \sim y$ on M as follows: two points x, y are equivalent if for any path γ connecting x to y and any harmonic form α representing an integer cohomology class, the number $\int_\gamma \alpha$ is integer. The quotient $M/\sim$ is called **the Albanese variety** of M, denoted by $\operatorname{Alb}(M)$.

Exercise 11.1. Let (M, ω) be a compact Kähler manifold, and v a field of holomorphic symplectomorphisms. Prove that v is Hamiltonian if and only if v acts trivially on $\operatorname{Alb}(M)$.

Exercise 11.2. Let S be a compact regular Sasakian manifold, $S/\mathsf{Reeb} = X$ the corresponding projective manifold.

a. Prove that a holomorphic vector field on X can be lifted to a Sasakian isometry of S if and only if it is Hamiltonian.

b. Let G_0 be a connected component of the group of Sasakian isometries of S, and $\operatorname{Ham}(X)$ the group of holomorphic Hamiltonian diffeomorphisms of X. Consitruct an exact sequence $0 \longrightarrow S^1 \longrightarrow G_0 \longrightarrow \operatorname{Ham}(X) \longrightarrow 0$.

Hint. Let L be a Hermitian line bundle such that S is its space of unit vectors, and $\phi(\xi) := |\xi|^2$ its Kähler potential. Prove that

$$\omega \lrcorner v = (dd^c \log \phi) \lrcorner v = \operatorname{Lie}_v d^c \log \phi - d\langle d^c \log \phi, v\rangle.$$

Exercise 11.3. Let S be a 3-dimensional Sasakian manifold not diffeomorphic to a sphere or its quotient by a finite group. Prove that S is quasiregular.

Exercise 11.4. Let A be a local ring of a singular point on a complex variety. Find an example of a singular point such that the Lie algebra of automorphisms of A has dimension at least 2 and contains two non-proportional contractions.

Exercise 11.5. Find an example of a compact CR-manifold, not isomorphic to a sphere, and admitting 2 non-proportional Sasakian metrics.

Hint. Use the previous exercise.

Exercise 11.6. Let A be a local ring of a singular point on a complex variety. Find an example of a singular point such that the Lie algebra of automorphisms of A contains a contraction which does not lie in its center.

Exercise 11.7. Find an example of a compact Sasakian manifold admitting a CR-automorphism which is not an isometry.

Hint. Use the previous exercise.