LCK manifolds 12: Morse-Novikov and Bott-Chern cohomology

Exercise 12.1. Find an example of compact LCK (non-Kähler) manifold admitting symplectic structure.

Exercise 12.2. Let M be an LCK manifold with potential, and $\phi: \tilde M \longrightarrow \mathbb{R}$ its potential. Prove that $\phi \neq 0$.

Exercise 12.3. Let M be a compact complex manifold, $H^2(M) = H^1(M) = 0$. Prove that $\dim H^{1,1}_{BC}(M) = 2\dim H^1(\mathcal{O}_M)$, where $H^{p,q}_{BC}(M)$ denotes the Bott-Chern cohomology.

Exercise 12.4. Assume that the standard map $\bigoplus_{p,q} H^{p,q}_{BC}(M) \longrightarrow H^*(M)$ to the de Rham cohomology is injective. Prove that it is surjective.

Exercise 12.5. Assume that the standard map $\bigoplus_{p,q} H^{p,q}_{BC}(M) \longrightarrow \bigoplus_{p,q} H^{p,q}_{\bar\partial}(M)$ to the Dolbeault cohomology is injective. Prove that it is surjective.

Exercise 12.6. Find an example of a compact locally conformally symplectic manifold (M, ω, θ) such that the cohomology class of θ is non-zero, and the Morse-Novikov class $[\omega]_{MN}$ vanishes.

Exercise 12.7. Let M be a Riemannian 4-manifold, and $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$ the decomposition on eigenspaces of the Hodge $*$-operator. Consider the map $d_+ : \Lambda^1(M) \longrightarrow \Lambda^+(M)$ obtained by projecting de Rham differential to $\Lambda^+(M)$. Compute symbols and prove that $\ker d^+/\operatorname{im} d$ is finite-dimensional.

Exercise 12.8. Let G be a compact Lie group, and ω a left-invariant locally conformally symplectic form, $d\omega = \omega \wedge \theta$. Prove that either $\theta = 0$ or $[\omega]_{MN} = 0$.