LCK manifolds 13: Conformal symplectomorphisms

Definition 13.1. A **locally conformally symplectic** (LCS) manifold is a manifold M, $\dim_{\mathbb{R}} M > 2$ equipped with a non-degenerate 2-form ω such that $d\omega = \omega \wedge \theta$, and θ is closed. **Conformal symplectomorphism** of an LCS manifold is a diffeomorphism which maps ω to $e^f \omega$. The group of conformal symplectomorphisms is denoted $\mathsf{SConf}(M)$. We consider ω as a symplectic form with coefficients in a flat line bundle L, called **the weight bundle** of M. The Lie algebra $\mathfrak{sconf}(M) \subset TM$ of all vector fields V such that e^{tv} lies in $\mathsf{SConf}(M)$ is called **the Lie algebra of conformally symplectic vector fields**. **Monodromy group** $\operatorname{Mon}(M)$ of an LCS manifold is monodromy group of the flat connection on L.

Exercise 13.1. Let (M, ω) be a LCS manifold, and $\nu \in \mathsf{SConf}(M)$. Prove that ν acts as a symplectic homothety on the symplectic cover $(\tilde M, \tilde\omega)$, $\nu^*\tilde\omega = \chi(\nu)\tilde\omega$. Prove that $\chi: \ \mathsf{SConf}(M) \longrightarrow \mathbb{R}^*$ is a group homomorphism.

Exercise 13.2. Prove that the weight bundle L of an LCS manifold is $\mathsf{SConf}(M)$-equivariant. Construct an $\mathsf{SConf}(M)$-equivariant symplectic structure on the space of non-zero vectors in $L \otimes \mathbb{C}$.

Exercise 13.3. Construct a character $\chi: \ \mathfrak{sconf}(M) \longrightarrow \mathbb{R}$ such that $e^{\chi(v)} = \chi(e^v)$. Describe it explicitly in terms of $\tilde M$.

Exercise 13.4. Suppose that an LCS manifold M admits a vector field $v \in \mathfrak{sconf}(M)$, $\chi(v) \neq 0$, such that $e^t v$ induces a circle action. Prove that the quotient $M/\langle e^{tv}\rangle$ is a contact orbifold.

Exercise 13.5. Let (M, ω, θ) be an LCS-action, and $v \in \mathfrak{sconf}(M)$ inducing a circle action $\rho(t) := e^{tv}$. Prove that $\chi(v) = \int_S \theta$, where $S = S^1$ is an orbit of ρ.

Exercise 13.6. Let M be an LCS-manifold, $v \in \mathfrak{sconf}(M)$ a vector field, $\chi(v) \neq 0$, and $\rho(t) = e^{tv}$ the corresponding diffeomorphism flow. Suppose that the closure $\overline{\mathsf{im}\,\rho}$ in the diffeomorphism group (with C^1-topology) is compact. Prove that $\operatorname{Mon}(M) = \mathbb{Z}$.