ANALYSIS 1: SMOOTH MANIFOLDS.

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In order to be approved, you should solve in every sheet either all problems with asterisks or all problems without asterisks. The problems with two asterisks are optional: k problems with two asterisks substitute $2k$ problems with one asterisk. The problems with (!) are obligatory for everybody.

1.1. Topological manifolds

Remark. Manifolds can be smooth (of a given "class of smoothness"), real analytic, or topological (continuous). These types of manifolds admit different definitions. One may specify a type if necessary, but usually it is clear from the context.

Definition 1.1. A **topological manifold** of dimension n is a topological space where every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Remark. Let G be a group acting on a set M. The **stabilizer** of $x \in M$ is the subgroup of all elements in G that fix x. An action is **free** if the stabilizer of every point is trivial.

Remark. Any action of a group on a topological space is continuous by default.

Problem 1.1. Suppose that a finite group G acts freely on a Hausdorff manifold M. Show that the quotient space M/G is a manifold.

Problem 1.2 (!). Construct an example of a finite group G acting non-freely on a manifold M such that M/G is not a manifold.

Problem 1.3. Consider the quotient of $\mathbb{R}^2$ by the action of $\{\pm 1\}$ that maps x to $-x$. Is the quotient space a manifold?

Problem 1.4 (*). Show that the n-dimensional sphere $\mathbb{S}^n$, the n-dimensional real projective space $\mathbb{P}^n_{\mathbb{R}}$, and the n-dimensional complex projective space $\mathbb{P}^n_{\mathbb{C}}$ are manifolds.

Remark. In the above definition of a manifold, we do not require it to be Hausdorff. Nevertheless, in many cases, manifolds are assumed to be Hausdorff by default.

Problem 1.5. Construct an example of a non-Hausdorff manifold.

Problem 1.6. Show that $\mathbb{R}^2/\mathbb{Z}^2$ is a manifold.

Problem 1.7. Let α be an irrational number. The group $\mathbb{Z}^2$ acts on $\mathbb{R}$ by the formula $t \mapsto t + m + n\alpha$. Show that this action is free, but the quotient $\mathbb{R}/\mathbb{Z}^2$ is not a manifold.

Problem 1.8 (**). Construct an example of a manifold of positive dimension such that the closures of two arbitrary nonempty sets always intersect or show that such a manifold cannot exist.

Problem 1.9 (**). Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a compact subgroup. Show that G is a manifold and that the quotient space $\mathrm{GL}(n, \mathbb{R})/G$ is also a manifold.

1.2. Smooth manifolds

Definition 1.2. A **cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every V_i is included in some U_i.

Problem 1.10. Show that any two cover of a topological space admit a common refinement.

Definition 1.3. A cover $\{U_i\}$ is an **atlas** if, for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ that provides a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. The **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are induced by the above homeomorphisms. An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class** C^i if they are i times differentiable, and *real analytic* if all transition maps are developable in Taylor's series at every point.

Definition 1.4. A **refinement** of an **atlas** is a refinement of the corresponding cover $V_i \subset U_i$ equipped with the maps $\varphi_i : V_i \to \mathbb{R}^n$ that are the restrictions of $\varphi_i : U_i \to \mathbb{R}^n$. Two atlases (U_i, φ_i) and (U_i, ψ_i) of class C^∞ or C^i (with the same cover) are **equivalent** in this class if, for all i, the map $\psi_i \circ \varphi_i^{-1}$ defined on the corresponding open subset in $\mathbb{R}^n$ belongs to the mentioned class. Two arbitrary atlases are **equivalent** if the corresponding cover possess a common refinement and the corresponding refinements of the atlases are equivalent.

Definition 1.5. A **smooth structure** on a manifold (of class C^∞ or C^i) is an atlas of class C^∞ or C^i considered up to the above equivalence. A **smooth manifold** is a topological manifold equipped with a smooth structure.

Remark. Terrible, is not it?

Problem 1.11 (*). Construct an example of two nonequivalent smooth structures on $\mathbb{R}^n$.

Definition 1.6. A **smooth function** on a manifold M is a function f whose restriction to the chart (U_i, φ_i) provides a smooth map $f \circ \varphi_i^{-1}$ defined on the open subset $\varphi_i(U_i) \subset \mathbb{R}^n$.

Remark. There are several ways to define a smooth manifold. The above way is canonical. It is not the most convenient one but you should know it. Two other ways (via sheaves of functions and via Whitney's theorem) are presented in these sheets.

Definition 1.7. A **pre-sheaf** of **functions** on a topological space M is given by the following data. For every open subset $U \subset M$, it is given a subring $\mathcal{F}(U) \subset F(U)$ in the ring $F(U)$ of all functions on U such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$. A pre-sheaf is called a **sheaf** if such subrings satisfy the following conditions. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ and let $f_i \in \mathcal{F}(U_i)$ be a family of functions defined on the members of the cover and satisfying the condition

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.

Problem 1.12 (!). Let $\mathcal{F}$ be a pre-sheaf of functions. Show that $\mathcal{F}$ is a sheaf if and only if, for every cover $\{U_i\}$ of an open subset $U \subset M$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact.

Remark. An **exact sequence** is a sequence of abelian groups and homomorphisms

$$\cdots \to A_1 \to A_2 \to A_3 \to \ldots$$

such that the kernel of every arrow coincides with the image of the previous one.

Problem 1.13. Show that the following spaces of functions on $\mathbb{R}^n$ are rings and define sheaves of functions.

a. The space of continuous functions
b. The space of infinitely smooth functions
c. The space of i times differentiable functions
d(*). The space of functions that are pointwise limits of sequences of continuous functions
e. The space of functions vanishing outside a set of measure 0

Problem 1.14. Show that the following spaces of functions on $\mathbb{R}^n$ are pre-sheaves but are not sheaves.

a. The space of constant functions
b. The space of limited functions
c. The space of functions vanishing outside a limited subset
d(*). The space of Lebesgue measurable functions with finite measure

Definition 1.8. A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} satisfy the above condition (i.e., are morphisms of ringed spaces).

Remark. Frequently, the term "ringed space" stands for a more general concept, where the "sheaf of functions" is an abstract "sheaf of rings," not necessarily a subsheaf in the sheaf of all functions on M. The above definition is simpler although not quite standard.

Problem 1.15. Let M, N be open subsets in $\mathbb{R}^n$ and let $\Psi : M \to N$ be a smooth map. Show that Ψ defines a morphism of spaces ringed by smooth functions.

Problem 1.16. Let M be a smooth manifold of some class and let $\mathcal{F}$ be the space of functions of this class. Show that $\mathcal{F}$ is a sheaf.

Problem 1.17 (!). Let M be a topological manifold and let (U_i, φ_i) and (V_j, ψ_j) be smooth structures on M. Show that these structures are equivalent if and only if the corresponding sheaves of smooth functions coincide.

Remark. The above problem implies that the following definition is equivalent to Definition 1.5.

Definition 1.9. Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold** of **class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions on $\mathbb{R}^n$ of the mentioned class.

Definition 1.10. A **coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{R}^n$.

Remark. In order to avoid a complicated notation, from now on, we assume all manifolds to be Hausdorff and infinitely smooth. The case of other classes can be considered in the same manner.

Problem 1.18. Let $(M, \mathcal{F})$ and $(N, \mathcal{F}')$ be manifolds and let $\Psi : M \to N$ be a continuous map. Show that the following conditions are equivalent.

(a) In local coordinates, Ψ is given by a smooth map
(b) Ψ is a morphism of ringed spaces.

Remark. An isomorphism of smooth manifolds is called a **diffeomorphism**. A diffeomorphism is a homeomorphism that maps smooth functions onto smooth ones.

Problem 1.19 (*). Let $\mathcal{F}$ be a pre-sheaf of functions on $\mathbb{R}^n$. Figure out a minimal sheaf that contains $\mathcal{F}$ in the following cases.

(a) Constant functions
(b) Functions vanishing outside a limited subset
(c) Limited functions

Problem 1.20 (*). Consider the ringed space $(\mathbb{R}^n, C^i)$ with i times differentiable functions. Describe all morphisms from $(\mathbb{R}^n, C^{i+1})$ to $(\mathbb{R}^n, C^i)$.

1.2. Embedded manifolds

Definition 1.11. A **closed embedding** $N \hookrightarrow M$ of topological spaces is a homeomorphism of N with its image that is closed in M.

Definition 1.12. Let M be a smooth manifold of dimension m and let $N \subset M$ be a subset. Then N is called an **embedded manifold** of dimension n and the map $N \hookrightarrow M$ is called a **smooth embedding** if, for every point $x \in N$, there is a neighborhood $U \subset M$ diffeomorphic to $\mathbb{R}^m$ such that the diffeomorphism maps $U \cap N$ onto a linear subspace of dimension n. If the image of N is closed in M, the map $N \hookrightarrow M$ is called a **closed embedding**.

Problem 1.21 (!). Let $(M, \mathcal{F})$ be a smooth manifold and let $N \subset M$ be an embedded submanifold. Consider the space $\mathcal{F}'(U)$ of functions on $U \subset N$ that are extendable to functions on M defined on some neighborhood of U.

a. Show that $\mathcal{F}'$ is a sheaf.
b. Show that this sheaf defines a smooth structure on N.
c. Show that the natural embedding $(N, \mathcal{F}') \to (M, \mathcal{F})$ is a morphism of manifolds.

Problem 1.22. Let N_1, N_2 be two manifolds and let $\varphi_i : N_i \to M$ be smooth embeddings. Suppose that the image of N_1 coincides with that of N_2. Show that N_1 and N_2 are isomorphic.

Remark. By the above problem, in order to define a smooth structure on N, it suffices to embed N into $\mathbb{R}^n$. As it will be clear in the next sheet, every manifold is embeddable into $\mathbb{R}^n$ (assuming reasonable conditions). Therefore, in place of a smooth manifold, we can use "manifolds that are smoothly embedded into $\mathbb{R}^n$."

Problem 1.23. Construct a smooth embedding of $\mathbb{R}^2/\mathbb{Z}^2$ into $\mathbb{R}^3$.

Problem 1.24 (*). Show that $\mathbb{P}^n_{\mathbb{R}}$ does not admit a smooth embedding into $\mathbb{R}^{n+1}$ for $n > 1$.

1.4. Partition of unity

Definition 1.13. A cover $\{U_i\}$ of a topological space M is called **locally finite** if every point in M possesses a neighborhood that intersects only a finite number of members of the cover.

Problem 1.25. Let $\{U_i\}$ be a locally finite cover of M such that every U_i is homeomorphic to $\mathbb{R}^n$. Show that the cover possesses a locally finite refinement $\{V_i\}$ such that the closure of every V_i is compact in M.

Hint. Cover every $U_i = \mathbb{R}^n$ by the balls of radius 1 centered at integer points.

Problem 1.26 (!). Given a locally finite cover $\{U_i\}$ of a manifold M such that every U_i is equipped with a homeomorphism $U_i \xrightarrow{\varphi_i} \mathbb{R}^n$ and has compact closure in M, show that there exists a collection of numbers $r_i > 0$ such that the $\varphi_i^{-1}(B_{r_i})$ form a cover of M, where B_{r_i} stands for the open ball of radius r_i centered at 0.

Problem 1.27 (!). Let M be a manifold admitting a locally finite cover by open subsets homeomorphic to $\mathbb{R}^n$. Show that the cover has a locally finite refinement $\{U_i\}$ such that every U_i can be equipped with a homeomorphism $U_i \xrightarrow{\varphi_i} \mathbb{R}^n$ and the inverse images $\varphi_i^{-1}(B_1)$ of unit balls cover M as well. Verify that we can take smooth φ_i's if M is equipped with a smooth structure.

Definition 1.14. Let M be a smooth manifold and let $\{U_i\}$ be a locally finite cover of M. A **partition** of **unity** subordinate to the cover $\{U_i\}$ is a family of smooth functions $f_i : M \to [0, 1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.

(a) Every function f_i vanishes outside U_i

(b) $\sum_i f_i = 1$

Remark. Note that the sum $\sum_i f_i$ is well defined because the cover U_i is locally finite.

Problem 1.28. Show that all derivatives of $e^{-\frac{1}{x^2}}$ at 0 vanish.

Problem 1.29. Define the following function λ on $\mathbb{R}^n$

$$\lambda(x) := \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Show that λ is smooth and that all its derivatives vanish at the points of the unit sphere.

Problem 1.30. Let $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$ be an atlas on a smooth manifold M. Consider the following function $\lambda_i : M \to [0,1]$

$$\lambda_i(m) := \begin{cases} \lambda\big(\varphi_i(m)\big) & \text{if } m \in U_i \\ 0 & \text{if } m \notin U_i \end{cases}$$

Show that λ_i is smooth.

Problem 1.31 (!). Let $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$ be a locally finite atlas on a manifold M such that $\varphi_i^{-1}(B_1)$ cover M as well (such an atlas was constructed in Problem 1.27). Consider the functions λ_i's constructed in the above problem. Show that $\sum_j \lambda_j$ vanishes nowhere and that the family of functions $\left\{f_i := \frac{\lambda_i}{\sum_j \lambda_j}\right\}$ provides a partition of unity on M.

Problem 1.32 (!). Show that every manifold with enumerable base of topology admits a partition of unity.

1.5. Whitney's theorem for compact manifolds

Definition 1.15. Define $\mathbb{R}^\infty$ as the union of all $\mathbb{R}^i$ embedded one into the other by the maps $(x_1, \dots, x_n) \hookrightarrow (x_1, \dots, x_n, 0)$

Problem 1.33 (*). Show that $\mathbb{R}^\infty$ is not locally compact.

Problem 1.34. Show that $\mathbb{R}^\infty$ is a topological abelian group (i.e., equipped with a continuous commutative group operation, namely, the addition).

Problem 1.35 (*). Consider the unit sphere $\mathbb{S}^\infty \subset \mathbb{R}^\infty$. Show that it is contractible.

Problem 1.36 (*). Is the corresponding projective space $\mathbb{P}^\infty_{\mathbb{R}} := \mathbb{S}^\infty/\{\pm 1\}$ contractible?

Problem 1.37. Let M be a smooth manifold, let $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$ be a locally finite atlas, and let $\{f_i\}$ be a partition of unity subordinated to the atlas and such that $f_i = 0$ outside some compact subset in U_i. Consider the following map $\Psi_i : M \to \mathbb{R}^{n+1}$

$$\Psi_i(m) := \begin{cases} \big(f_i(m)\varphi_i(m), f_i(m)\big) & \text{if } m \in U_i \\ (0, \dots, 0) & \text{if } m \notin U_i \end{cases}$$

a. Show that Ψ_i is injective on the set where $f_i \neq 0$.

b. Suppose the the atlas $\{U_i\}$ is finite and contains m charts. Show that $\bigoplus_i \Psi_i$ defines a closed embedding of M into $\mathbb{R}^{(n+1)m}$.

c(*). Show that $\bigoplus_i \Psi_i$ defines a closed embedding of M into $\mathbb{R}^\infty$ if the number of charts in the atlas $\{U_i\}$ is infinite.

Problem 1.38 (!). Prove Whitney's theorem (for compact manifolds) : every compact manifold admits a closed smooth embedding into $\mathbb{R}^n$.

Problem 1.39 (*). Let $U \subset M$ be an open subset in a smooth manifold. Suppose that U is homeomorphic to $\mathbb{R}^n$ and let $V \subset U$ correspond to the unit ball. Construct a smooth map from M onto the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ which is injective on V and maps into $(0, \dots, 0, 1)$ the complement to U_i.

Problem 1.40 (**). Show that the map constructed in the above problem is necessarily surjective.