ANALYSIS 2: HAUSDORFF DIMENSION AND WHITNEY'S THEOREM.

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In order to be approved, you should solve in every sheet either all problems with asterisks or all problems without asterisks. The problems with two asterisks are optional: k problems with two asterisks substitute $2k$ problems with one asterisk. The problems with (!) are obligatory for everybody.

2.1. Hausdorff dimension

Definition 2.1. Let M be a metric space. The **diameter** $\operatorname{diam} M \in [0,\infty]$ is the number $\sup\limits_{x,y\in M} d(x,y)$.

Definition 2.2. In a metric space, the **ball** of radius ε centered at x is defined as the set of all points y satisfying $d(x,y) < \varepsilon$.

Problem 2.1. Describe all possible values of the diameter of the ball of radius ε in a metric space.

Problem 2.2. Let M be a metric space and let $\varepsilon > 0$. Show that M possesses a cover by balls of diameter $\leq \varepsilon$.

Definition 2.3. Let $\{S_i\}$ be a cover of a metric space M formed by balls of radius r with $r < \varepsilon$. Define $\mu_{d,\varepsilon} \in [0,\infty]$ as

$$\mu_{d,\varepsilon} M := \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^d,$$

where the infimum is taken with respect to all covers as above. The limit

$$\mu_d M := \sup \lim_{\varepsilon\to 0} \mu_{d,\varepsilon} M$$

is called d-**dimensional Hausdorff measure** of M.

Problem 2.3. Suppose that a metric in $M = \mathbb{R}^n$ is given by the norm $\big|(x_1,\ldots,x_n)\big| := \max|x_i|$. Show that the n-dimensional Hausdorff measure of a polyhedron equals its volume (in the usual sense).

Problem 2.4 (*). Suppose that a metric in $M = \mathbb{R}^n$ is given by the norm $\big|(x_1,\ldots,x_n)\big| := \sum|x_i|$. Show that the n-dimensional Hausdorff measure of a polyhedron is proportional to its volume. Calculate the coefficient of proportionality.

Problem 2.5 (*). Let $M = \mathbb{R}^n$ be equipped with the euclidean metric. Show that the n-dimensional Hausdorff measure of a polyhedron is proportional to its volume. Calculate the coefficient of proportionality.

Definition 2.4. A map $f : M \to N$ of metric spaces is called lipschitz with constant $C \geq 0$ if $d(x,y) \geq C \cdot d\big(f(x), f(y)\big)$ for all $x, y \in M$. A map is called bilipschitz if it is bijective and the inverse map is also lipschitz (with some constant).

Problem 2.6. Show that every lipschitz map is continuous.

Problem 2.7 (*). Construct an example of a continuous map of metric spaces that is not lipschitz.

Problem 2.8. Let d_1, d_2 be two norms on a vector space V. Denote by the same letters the corresponding metrics. Show that the identity map $\operatorname{Id}_V : (V, d_1) \to (V, d_2)$ is lipschitz if and only if the unit ball $B_1(r, d_1)$ is limited in terms of the norm d_2.

Problem 2.9 (*). Let $M = \mathbb{R}^n$ and let d_1, d_2 be some norms on M. Show that $\operatorname{Id}_M : (M, d_1) \to (M, d_2)$ is bilipschitz.

Problem 2.10 (!). Let $U \subset \mathbb{R}^n$ be a limited open subset and let $\Phi : U \to \mathbb{R}^n$ be a smooth map smoothly extendable to the boundary ∂U. Show that Φ is lipschitz.

Problem 2.11. Let $M \xrightarrow{f} N$ be a lipschitz map of metric spaces with constant C. Show that $\mu_d M \geq C^d \mu_d f(M)$, where μ_d stands for the d-dimensional Hausdorff measure.

Problem 2.12 (!). Suppose that $\mu_d M < \infty$. Show that $\mu_{d'} M = 0$ for every $d' > d$.

Hint. Deduce from $\operatorname{diam} S_i < \varepsilon$ the inequality

$$\mu_{d',\varepsilon} M = \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^{d'} \leq \varepsilon^{d'-d} \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^d = \varepsilon^{d'-d} \mu_{d,\varepsilon} M \qquad (1)$$

and pass to the limit $\varepsilon \to 0$.

Problem 2.13 (!). Suppose that $\mu_{d'} M = \infty$. Show that $\mu_d M = 0$ for every $d < d'$.

Hint. Use the inequality (1) and pass to the limit $\varepsilon \to 0$.

Definition 2.5. Let M be a metric space. The **Hausdorff dimension** $\dim_H M \in [0, \infty]$ is the supremum of all d such that $\mu_d M = \infty$.

Problem 2.14. Find the Hausdorff dimension of a finite set.

Problem 2.15. Let $f : M \to N$ be a lipschitz map. Show that f does not increase the Hausdorff dimension: $\dim_H M \geq \dim_H f(M)$.

Problem 2.16. Show that every bilipschitz map preserve Hausdorff dimension ("Hausdorff dimension is a bilipschitz invariant").

Problem 2.17 (*). Find the Hausdorff dimension of the Cantor set $K \subset [0, 1]$.

Definition 2.6. A subset $Z \subset \mathbb{R}^n$ has **measure zero** if, for every $\varepsilon > 0$, there exists a countable cover of Z by balls U_i such that $\sum_i \operatorname{Vol} U_i < \varepsilon$.

Problem 2.18. Show that the countable union of subsets of zero measure has measure zero.

Problem 2.19. Show that the image of a subset of zero measure under a lipschitz map $\mathbb{R}^n \to \mathbb{R}^n$ has measure zero.

Problem 2.20 (!). Show that the image of a subset of zero measure under a smooth map $\mathbb{R}^n \to \mathbb{R}^n$ has measure zero.

Problem 2.21 (!). Construct an example of a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that sends a subset of zero measure onto a subset of nonzero measure.

Problem 2.22 (!). Let $M \subset \mathbb{R}^d$ be a subset such that $\dim_H M < d$. Show that M has measure zero.

Definition 2.7. Let M be a smooth manifold with an atlas $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$. A subset $Z \subset M$ has **measure zero** if the image $\varphi(Z \cap U_i)$ has measure zero in $\mathbb{R}^n$ for every i.

Problem 2.23. Show that this definition does not depend on the choice of an atlas on M.

Problem 2.24. Let $M \xrightarrow{f} \mathbb{R}^n$ be a smooth map of manifolds and let M be a union of compact subsets. Show that $\dim_H f(M) \leq \dim M$.

Hint. Show first that f is lipschitz on compact subsets. Then use the fact that lipschitz maps satisfy $\dim_H f(M) \leq \dim M$.

Problem 2.25 (!). Let $M \xrightarrow{f} N$ be a smooth map of manifolds such that $\dim M < \dim N$. Show that the image of M has measure zero.

Hint. Use the previous problem.

Remark. This theorem is a particular case of Sard's theorem that claims that the set of critical values of a smooth map has measure zero.

Problem 2.26 (**). Deduce Sard's theorem from the above problem.

2.2. Whitney's theorem (with a bound on dimension)

Definition 2.8. A smooth map of manifolds $M \xrightarrow{f} N$ is called **immersion** if the differential $\mathcal{D}f$ is an embedding in local coordinates.

Definition 2.9. The **Klein bottle** is the quotient of the two-dimensional torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$ by the action of the group $\mathbb{Z}/2\mathbb{Z}$ mapping (t_1, t_2) to $(t_1 + \pi, -t_2)$.

Problem 2.27. Show that the indicated action is free and that the quotient is a manifold.

Problem 2.28. Construct an immersion of the Klein bottle into $\mathbb{R}^3$.

Problem 2.29 (!). Let $M \xrightarrow{f} N$ be a smooth map of manifolds. Show that f is a smooth embedding if and only if it is an injective immersion.

Hint. Use the inverse function theorem.

Definition 2.10. Let $M \hookrightarrow \mathbb{R}^n$ be a smooth m-submanifold. The **tangent plane** at $p \in M$ is the plane in $\mathbb{R}^n$ tangent to M (i.e, the plane lying in the image of the differential given in local coordinates). A **tangent vector** is an arbitrary vector in this plane with the origin at p. The space of all tangent vectors at p is denoted by $\mathrm{T}_p M$. When a metric on $\mathbb{R}^n$ is given, we can define the space of **unit tangent vectors** $\mathbb{S}^{m-1}M$ as the set of all pairs (p, v), where $p \in M$, $v \in \mathrm{T}_p M$, and $|v| = 1$.

Problem 2.30. Show that $\mathbb{S}^{m-1}M$ is a manifold and that the natural projection $\mathbb{S}^{m-1}M \to M$ is a smooth map with fibers $\mathbb{S}^{m-1}$.

Remark. $\mathbb{S}^{m-1}M$ is called the **unit sphere bundle** over M.

Problem 2.31 (*). Show that $\mathbb{S}^{m-1}M$ is independent of an embedding $M \hookrightarrow \mathbb{R}^n$, i.e., for two different embeddings of M into $\mathbb{R}^n$ and into $\mathbb{R}^{n'}$, the corresponding manifolds $\mathbb{S}^{m-1}M$ are diffeomorphic.

Problem 2.32 (!). Let $M \overset{\varphi}{\hookrightarrow} \mathbb{R}^n$ be a manifold of dimension m embedded into $\mathbb{R}^n$, let $\lambda \in \mathbb{P}^{n-1}_{\mathbb{R}}$ be a straight line in $\mathbb{R}^n$, and let $P_\lambda : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection onto the quotient $\mathbb{R}^n/\lambda \cong \mathbb{R}^{n-1}$.

a. Let $\Delta \subset M \times M$ stand for the diagonal. Define the map $M \times M \setminus \Delta \xrightarrow{B} \mathbb{P}^{n-1}_{\mathbb{R}}$ by sending the pair of points $(x, y) \in M \times M$ to the straight line passing through $\varphi(x) - \varphi(y)$. Show that $P_\lambda \circ \varphi : M \to \mathbb{R}^{n-1}$ is an injection if and only if λ does not lie in the image of B.

b. Define the map $\mathbb{S}^{m-1}M \xrightarrow{B_0} \mathbb{P}^{n-1}_{\mathbb{R}}$ by sending a tangent vector to the corresponding straight line. Show that $P_\lambda \circ \varphi : M \to \mathbb{R}^{n-1}$ is an immersion if and only if λ does not lie in the image of B_0.

Problem 2.33 (!). Let $M \overset{\varphi}{\hookrightarrow} \mathbb{R}^n$ be an embedded manifold of dimension m with $n > 2m + 2$. Show that there exists a projection $\mathbb{R}^n \xrightarrow{P} \mathbb{R}^{2m+2}$ such that $P \circ \varphi : M \to \mathbb{R}^{2m+2}$ is an immersion.

Hint. Use the fact that the images of the maps B_0 and B in the previous problem have measure zero and apply induction on n.

Problem 2.34. Under the conditions of the previous problem, show that there exists a projection $\mathbb{R}^n \xrightarrow{P} \mathbb{R}^{2m+1}$ such that $P \circ \varphi : M \to \mathbb{R}^{2m+1}$ is an immersion.

Problem 2.35. Is any n-dimensional manifold embeddable in $\mathbb{R}^{2n-1}$?

Problem 2.36 (**). Is it possible to construct an immersion of the projective space $\mathbb{P}^2_{\mathbb{C}}$ into $\mathbb{R}^5$?

Problem 2.37. Let M be a compact Hausdorff manifold of dimension n. Show that M admits a smooth closed embedding into $\mathbb{R}^{2n+2}$.

Remark. Whitney showed that any Hausdorff m-dimensional manifold with a countable basis of topology admits a closed embedding into $\mathbb{R}^{2m}$. This statement is called the "strong Whitney theorem."

2.3. Whitney's theorem (for noncompact manifolds)

Problem 2.38. Let $\mathcal{P}$ denote the space of embeddings $\mathbb{R}^m \to \mathbb{R}^{2m+2}$ equipped with the natural topology. Show that $\mathcal{P}$ is a manifold. Construct a smooth structure on $\mathcal{P}$.

Problem 2.39. Let M be a manifold with a countable basis.

a. Show that M is a union of an ascending countable chain of compact subsets.
b. Show that M admits a partition of unity.

Problem 2.40. Let M be an n-dimensional manifold, let $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$ be a locally finite atlas, and let $f_i : U_i \to [0,1]$ be a corresponding partition of unity. Consider the map $\Psi_i : M \to \mathbb{R}^{n+1}$ constructed as in the previous sheet

$$\Psi_i(m) := \begin{cases} \big(f_i(m)\varphi_i(m), f_i(m)\big) & \text{if } m \in U_i \\ (0, \dots, 0) & \text{if } m \notin U_i. \end{cases}$$

Let $A_i \in \mathcal{P}$ be a family of embeddings $\mathbb{R}^n \to \mathbb{R}^{2n+2}$ with the same set of indices. Consider the map $\Psi_A : M \to \mathbb{R}^{2n+2}$, $\Psi_A(m) := \sum A_i\big(\Psi_i(m)\big)$. Show that this map is well defined. Show that it can be obtained as a composition of the embedding $\bigoplus_i \Psi_i : M \to \mathbb{R}^\infty$ and a linear projection $\mathbb{R}^\infty \to \mathbb{R}^{2n+2}$.

Problem 2.41 (*). Under the conditions of the previous problem, let $M_0 \subset M$ be a compact subset and let $\bigcup_{i\in I} U_i \supset M_0$ be a corresponding finite subcover in $\{U_i\}$ of k members. Show that there exists a subset $Z_I \subset \mathcal{P}^k$ of zero measure such that, for all collections $\{A_i,\ i \in I\} \in \mathcal{P}^k$ not belonging to Z_I, the corresponding map $\Psi_A : M_0 \to \mathbb{R}^{2n+2}$ is a smooth embedding.

Hint. Use the proof of the Whitney theorem for compact M given in the previous section.

Problem 2.42 (*). Denote by $\mathcal{P}^\infty$ the product of $\mathcal{P}$ with respect to the same set of indices as the one used in the atlas $\{U_i\}$. Consider $\mathcal{P}^\infty$ equipped with the product Lebesgue measure. Show that the set Z of all $\{A_i\} \in \mathcal{P}$ such that Ψ_A is not an embedding has measure zero in $\mathcal{P}^\infty$.

Hint. By construction, Z is the union of all inverse images of the sets $Z_I \subset \mathcal{P}^k$ constructed in Problem 2.41 under the standard projection $\mathcal{P}^\infty \xrightarrow{\Pi_I} \mathcal{P}^k$. Every such inverse image has measure zero, hence, Z, being a union of subsets of zero measure, has measure zero.