Symplectic geometry: exam

Handouts score is given by the formula $t = 10a + 5b + 5c$, where a is the number of completed handouts with at least 2/3 exercises credited within 3 weeks since the exercise was distributed, b the number of handouts with 1/3 exercises credited within 3 weeks, and c the number of handouts with 2/3 exercises credited, if it is not already counted with a and b.

Each student receives a random selection of 12 test problems, 3 from each section (the output of the randomizer is printed on a separate sheet). The final score for the course is $s = 2+p+[t/10]$, where p is the total number of points for the exam. The exam is oral.

1 Symplectic forms

Exercise 1.1. Construct two symplectic forms ω_1, ω_2 on $\mathbb{R}^4$ such that $\omega_1 + \omega_2$ has non-constant rank.

Exercise 1.2. Let S be a 2-dimensional smooth manifold, and ω_1, ω_2 two symplectic forms in the same homotopy class. Prove that $\omega_1+\omega_2$ is symplectic. Find an example of symplectic forms ω_1, ω_2 on $\mathbb{R}^2$ such that $\omega_1+\omega_2$ has non-constant rank.

Exercise 1.3. Let $U(n), O(2n), Sp(2n), GL(n, \mathbb{C})$ be classical Lie groups embedded to $GL(2n, \mathbb{R})$ in the usual way. Prove that $O(2n) \cap Sp(2n) = O(2n) \cap GL(n, \mathbb{C}) = Sp(2n) \cap GL(n, \mathbb{C}) = U(n)$.

Exercise 1.4 (2 points). Prove that $\pi_1(Sp(2n)) = \mathbb{Z}$.

Exercise 1.5 (2 points). Let $V = \mathbb{R}^{2n}$ be a symplectic vector space, and X the Grassmanian of oriented Lagrangian subspaces in V. Prove that $\pi_1(X)$ is infinite.

Exercise 1.6 (2 points). Let U be the set of non-degenerate bilinear antisymmetric 2-forms on $\mathbb{R}^{2n}$. Prove that U has precisely 2 connected components.

Exercise 1.7. Prove that the group $Sp(2n)$ of symplectic matrices acts transitively on the set of $n + k$-dimensional coisotropic subspaces.

Exercise 1.8. Let M be a compact symplectic manifold, and $U \subset H^2(M, \mathbb{R})$ the set of all cohomology classes which are represented by a symplectic form. Prove that U is open in $H^2(M, \mathbb{R})$.

Exercise 1.9 (2 points). Let M be a compact oriented manifold which is smoothly fibered over a symplectic manifold X with symplectic fibers. Prove that M admits a symplectic structure or find a counterexample.

Exercise 1.10. Let $M = (S^1)^n \times S^3$. Prove that M does not admit a symplectic structure for any n.

2 Symplectomorphisms

Definition 2.1. An antisymplectic diffeomorphism is a diffeomorphism Ψ of a symplectic manifold (M, ω) such that $\Psi^*\omega = -\omega$.

Exercise 2.1 (2 points). Find a compact symplectic manifold (M, ω) admitting an antisymplectic diffeomorphism. Find a compact symplectic manifold which does not admit an antisymplectic diffeomorphism.

Exercise 2.2 (3 points). Let Ψ be a diffeomorphism of a connected symplectic manifold (M, ω) which satisfies $\Psi^*\omega = 2\omega$. Prove that ω is exact, or find a counterexample.

Exercise 2.3. Let $\omega_i \in \Lambda^2(M)$ be a sequence of symplectic forms on a compact manifold M converging to a symplectic form ω in C^0-topology. Assume that all ω_i are homologous. Prove that almost all (M, ω_i) are symplectomorphic.

Exercise 2.4. Let (M, I) be a compact almost complex manifold, and $\omega_1, \omega_2 \in \Lambda^2(M)$ be almost Kähler forms. Assume that ω_1 is homologous to ω_2. Prove that (M, ω_1) is symplectomorphic to (M, ω_2).

Exercise 2.5. Let $M = \mathbb{C}^*$ be equipped with the standard symplectic form $\omega = dx \wedge dy$.

a. Find a symplectomorphism mapping a circle of radius 1 to a circle of radius 2, or prove it does not exist.

b. Construct a symplectomorphism mapping M to T^*S^1 with the standard symplectic structure, or prove it does not exist.

Exercise 2.6. Let $S \subset S^2$ be an equator, that is, a geodesic circle, and $\phi : S^2 \longrightarrow S^2$ a symplectomorphism. Prove that $\phi(S) \cap S \neq \emptyset$.

Exercise 2.7. Prove that the group of symplectomorphisms of (M, ω) acts transitively on M, for any connected symplectic manifold (M, ω).

3 Lagrangian subvarieties

Exercise 3.1. Construct a Lagrangian torus in $\mathbb{C}P^2$ with the standard symplectic structure.

Exercise 3.2. Let $Z \subset M$ be a submanifold. Define **the conormal bundle** CZ as the space of all pairs $z, v \in T^*M$ where $z \in Z$ and the form $v \in T^*_z M$ vanishes on $T_z Z \subset T_z M$. Prove that CZ is a Lagrangian submanifold in T^*M with the standard symplectic structure.

Exercise 3.3. Let $\pi : \ T^*M \longrightarrow M$ be the standard projection, and L_1, L_2 two Lagrangian sections of π (that is, Lagrangian submanifolds such that $\pi : L_i \longrightarrow M$ is a diffeomorphism). Assume that $h^1(M) = 0$. Prove that $L_1 \cap L_2 \neq \emptyset$.

Exercise 3.4. Let $M = T^2$ and let $\phi : \ t \longrightarrow 2t$ be the standard 4:1 covering. Construct a symplectic structure on $M \times M$ such that the graph of ϕ is Lagrangian.

Exercise 3.5. Let $M = M_1 \times M_2$, where $M_1, M_2 \cong T^2$. Find a symplectic structure on M such that the homology class $[M_1] + n[M_2]$ can be represented by a Lagrangian submanifold, for any given $n \in \mathbb{Z}^{>0}$.

Exercise 3.6 (2 points). Prove that $M = T^*S^2$ admits a symplectic structure ω such that all Lagrangian submanifolds $L \subset M$ diffeomorphic to S^2 are contractible in M.

Exercise 3.7 (2 points). Let (M, ω) be a $2n$-dimensional symplectic manifold, and $M \longrightarrow X$ a smooth submersion with Lagrangian fibers diffeomorphic to $\mathbb{R}^n$. Prove that M is symplectomorphic to T^*X.

Remark 3.1. Let η be a 1-form on a manifold M. **The zero set** of η is the set of all points $x \in M$ such that $\eta\Big|_{T_xM} = 0$.

Exercise 3.8. Let $X \subset (M, \omega)$ be a Lagrangian submanifold. Prove that there is a 1-form η defined in a neighbourhood U of X such that X is the zero set of η and $\omega\Big|_U = d\eta$.

Exercise 3.9. Let η be a 1-form on M, $\dim_{\mathbb{R}} M = 2n$, such that the 2-form $\omega := d\eta$ is symplectic. Assume that the zero set of η is an n-dimensional submanifold of M. Prove that it is ω-Lagrangian.

Exercise 3.10 (3 points). Construct an exact symplectic form on $\mathbb{C}P^3 \backslash \mathbb{C}P^1$.

4 Hamiltonians

Exercise 4.1. Let X be a Hamiltonian vector field on a symplectic manifold (M, ω), and e^{tX} the corresponding diffeomorphism flow. Prove that $(e^{tX})^*H = H$, where H is the Hamiltonian of X.

Exercise 4.2. Let X_1, X_2 be Hamiltonian vector fields on (M_1, ω_1) and (M_2, ω_2). Prove that $X_1 + X_2$ is a Hamiltonian vector field on $M_1 \times M_2$.

Exercise 4.3. Let (M, ω) be a symplectic manifold, and $F = (f_1, ..., f_n)$ a collection of functions such that the differentials df_i are linearly independent, and the corresponding Poisson brackets vanish: $\{f_i, f_j\} = 0$, for all i, j.

a. Prove that $2n \leqslant \dim_{\mathbb{R}} M$.

b. Suppose that $2n = \dim_{\mathbb{R}} M$. Prove that the level sets $F^{-1}(c)$ are Lagrangian for any $c \in \mathbb{R}^n$.

Exercise 4.4. Let $X \in TM$ be a vector field on M, and e^{tX} the corresponding diffeomorphism flow. Prove that its action on T^*M is always Hamiltonian.

Exercise 4.5. Construct a Hamiltonian vector field with non-closed orbits on a 4-dimensional torus equipped with the standard symplectic structure.

Exercise 4.6. Let $T^*X \xrightarrow{\pi} X$ be the Lagrangian fibration on T^*X with the standard symplectic structure. Consider the group G generated by Hamiltonian symplectomorphisms associated with functions $f \in \pi^* C^\infty X$. Prove that G is commutative and acts transitively on the set of Lagrangian sections of π, if $H^1(X) = 0$.

Exercise 4.7. Let G be a compact group freely acting on a manifold M. Prove that the induced action on T^*M admits an equivariant moment map.

Exercise 4.8 (2 points). Let G be a compact group freely acting on a manifold M. Prove that $T^*M /\!\!/ G = T^*(M/G)$.

Exercise 4.9 (2 points). Let $G = SO(3)$ act on S^2 by standard rotations. Denote by μ the natural embedding $\mu: S^2 \longrightarrow \mathbb{R}^3$. Prove that after an appropriate identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$, the moment map can be identified with μ.

Exercise 4.10. Let $S \subset M$ be a smooth hypersurface in a symplectic manifold M, given as a level set of a function H,

$$S = \{x \in M \ \mid H(x) = 0\}.$$

Prove that the corresponding Hamiltonian vector field X_H preserves S, and is tangent to the characteristic foliation on S.