# Symplectic handout 1: Non-degenerate 2-forms

**Definition 1.1.** Let $V$ be a vector space. A **complex structure** on $V$ is an operator $I \in \operatorname{End}(V)$ which satisfies $I^2 = -\operatorname{\mathsf{Id}}$.

**Exercise 1.1.** Let $V$ be a real vector space, and $X$ the set of complex subspaces $W \subset V \otimes_{\mathbb{R}} \mathbb{C}$ which satisfy $W \cap \bar{W} = 0$, $W + \bar{W} = V \otimes_{\mathbb{R}} \mathbb{C}$. Construct a $GL(V)$-invariant bijective correspondence between $X$ and the space of complex structures on $V$.

**Definition 1.2.** Let $M$ be a manifold. An endomorphism $I \in \operatorname{End}(TM)$, $I^2 = -\operatorname{\mathsf{Id}}_{TM}$ is called **an almost complex structure**. An $I$-invariant Riemannian metric is called **an Hermitian metric**.

**Exercise 1.2.** Let $(M, I)$ be an almost complex manifold, $\dim_{\mathbb{C}} M = n$. Prove that $(M, I)$ always admits a Hermitian metric $g$. Consider the orientation form $\omega^n$, obtained as the top exterior power of the corresponding Hermitian form $\omega$. Prove that the orientation defined by $\omega^n$ is independent from the choice of $g$.

**Exercise 1.3.** Let $g$ be a positive definite scalar product on a vector space $V$.

a. Construct a bijection between bilinear symmetric forms on $V$ and operators $A \in \operatorname{End}(V)$ satisfying $g(A(x), y) = g(x, A(y))$ (such operators are called **symmetric**, or **self-adjoint**).

b. Construct a bijection between bilinear anti-symmetric forms on $V$ and operators $A \in \operatorname{End}(V)$ satisfying $g(A(x), y) = -g(x, A(y))$ (such operators are called **anti-symmetric**).

c. A symmetric matrix $A$ is called **positive** if the bilinear symmetric form $x, y \longrightarrow g(A(x), y)$ is positive definite. Construct a bijection between positive symmetric matrices and positive definite bilinear symmetric forms.

d. Let $A$ be a non-degenerate anti-symmetric operator. Prove that $-A^2$ is positive symmetric.

**Exercise 1.4.** Let $g, g_1$ be bilinear symmetric forms on $V$, with $g$ positive definite and $S_g \subset V$ the sphere $\{v \in V \mid g(v,v) = 1\}$.

a. Denote by $x \in S_g$ an extremum of the function $x \longrightarrow g_1(x,x)$ on $S_g$. Prove that the 1-forms $g(x, \cdot)$ and $g_1(x, \cdot)$ are proportional.

b. ("Simultaneous diagonalization theorem") Prove that there exists an orthonormal basis $x_1, ..., x_n$ in $V$ such that $g_1(x_i, x_j) = 0$ for $i \neq j$.

**Hint.** Take for $x_1$ the point of $S_g$ where $x \longrightarrow g_1(x,x)$ reaches maximum, pass to $x^\perp$ and apply induction on $\dim V$.

**Exercise 1.5.** Let $A$ be a positive symmetric operator, and $A_1$ a positive symmetric operator satisfying $A_1^2 = A$. Prove that $A_1$ is unique.

**Hint.** Use the simultaneous diagonalization theorem.

**Exercise 1.6.** Let $\omega$ be an antisymmetric 2-form on a vector space $V = \mathbb{R}^{2n}$, and $g$ a positive definite scalar product. Prove that there exists a basis $x_1, ..., x_{2n}$, orthonormal with respect to $g$, such that $\omega$ is written in this basis as

$$\begin{pmatrix} 0 & a_1 & 0 & 0 & ... & 0 & 0 \\ -a_1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & a_2 & ... & 0 & 0 \\ 0 & 0 & -a_2 & 0 & ... & 0 & 0 \\ ... \\ 0 & 0 & 0 & 0 & ... & 0 & a_n \\ 0 & 0 & 0 & 0 & ... & -a_n & 0 \end{pmatrix}$$

where $a_1, ..., a_n$ are non-negative real numbers.

**Hint.** Apply the simultaneous diagonalization theorem to $g, -A^2$, where $A$ is the anti-symmetric operator corresponding to $\omega$.

**Exercise 1.7.** Let $(M, g)$ be a Riemannian manifold, and $A \in \operatorname{End}(TM)$ a positive symmetric operator, smooth in $M$. Prove that there exists a positive symmetric operator $A_1 \in \operatorname{End}(TM)$ such that $A_1^2 = A$. Prove that $A_1$ is smooth.

**Hint.** Use the uniqueness of $A_1$.

**Exercise 1.8.** Let $g$ be a positive definite form on a manifold, $\omega$ a non-degenerate 2-form, and $A \in \operatorname{End}(TM)$ the corresponding anti-symmetric operator.

a. Prove that there exists a positive symmetric operator $A_1 \in \operatorname{End}(TM)$, smooth in $M$, such that $A_1^2 = -A^2$.

b. Consider the symmetric form $g_1(x, y) := g(A_1(x), y)$, and let $I$ be an operator which satisfies $g_1(I(x), y) = \omega(x, y)$. Prove that $I^2 = -\operatorname{\mathsf{Id}}$.

c. Let $M$ be a manifold admitting a non-degenerate 2-form. Prove that $M$ admits an almost complex structure.

**Exercise 1.9 (*).** Prove that the space of almost complex structures on $M$ is homotopy equivalent to the space of non-degenerate 2-forms on $M$.

**Exercise 1.10 (*).** Let $\Omega$ be a non-degenerate complex linear 3-form on $\mathbb{C}^3$, and $\rho := \operatorname{Re}(\Omega)$ the corresponding form on $\mathbb{R}^6$. Let $\widetilde{SL}(3, \mathbb{C})$ be the group generated by $SL(2, \mathbb{C})$ and the complex conjugation. Prove that the group of all $A \in GL(6, \mathbb{R})$ such that $A(\rho) = \rho$ is isomorphic to $\widetilde{SL}(3, \mathbb{C})$.