Symplectic handout 2: diffeomorphism flows

Exercise 2.1. Prove that each $\mathbb{R}$-linear derivation of the ring $C^\infty\mathbb{R}^n$ is induced by a vector field on $\mathbb{R}^n$.

Exercise 2.2 (*). Let M be a manifold. Prove that each $\mathbb{R}$-linear derivation of the ring $C^\infty M$ is induced by a vector field on M.

Exercise 2.3. Let M be a compact manifold, and $\mathfrak{m}$ a maximal ideal in the ring $C^\infty M$ of all smooth real functions. Prove that $\mathfrak{m}$ is the ideal of all functions vanishing in a point of M.

Remark 2.1. We denote the set of maximal ideals of $C^\infty M$ by $\mathsf{Spec}_{\mathbb{R}}(C^\infty M)$. This set is identified with M; we induce the topology on $\mathsf{Spec}_{\mathbb{R}}(C^\infty M) = M$ from M.

Exercise 2.4. Let M be a compact manifold and A an $\mathbb{R}$-linear automorphism of $C^\infty M$. Prove that the corresponding map $A: \mathsf{Spec}_{\mathbb{R}}(C^\infty M) \longrightarrow \mathsf{Spec}_{\mathbb{R}}(C^\infty M)$ is a homeomorphism with respect to the topology induced from M.

Exercise 2.5. Let M be a compact manifold, and $\operatorname{Aut}_{\mathbb{R}}(C^\infty M)$ the group of all $\mathbb{R}$-linear automorphisms of the ring $C^\infty M$. Prove that the natural embedding from $\operatorname{Diff}(M)$ to $\operatorname{Aut}_{\mathbb{R}}(C^\infty M)$ is an isomorphism.

Hint. Use the previous exercise.

Definition 2.1. Let $v_t \in TM$ be a vector field, depending on a parameter $t \in [0,a]$ and $\Psi_t \in \operatorname{Diff}(M)$ a flow of diffeomorphisms, $t \in [0,a]$. We say that Ψ_t is **tangent to** v_t, or **is obtained by integrating** v_t, or **is equal to exponent of** v_t if for all $m \in M$, the vector $\Psi_t^{-1}\frac{d\Psi_t}{dt} \in T_m M$ is equal to $v_t\big|_m$.

Exercise 2.6. Let $v_t = v \in TM$ be a time-independent vector field, and $\Psi_t \in \operatorname{Diff}(M)$ the corresponding flow of automorphisms. Prove that v is Ψ_t-invariant.

Exercise 2.7. Find a vector field v on $M = \mathbb{R}$ such that there is no flow of diffeomorphisms $\Psi_t \in \operatorname{Diff}(M)$, $t \in [0,\varepsilon]$ tangent to v for any $\varepsilon > 0$.

From now on, the manifold M is assumed to be equipped with a finite covering $\{U_i\}$ by open balls, with all successive intersection of balls diffeomorphic to open balls or empty. We start with proving Theorem 1 from Lecture 2.

Theorem 1: Let $\alpha_t \in \Lambda^k(M)$, $t \in [0,1]$ be a smooth family of exact forms on M. Then there exists a smooth family of forms $\eta_t \in \Lambda^{k-1}(M)$, $t \in [0,1]$, such that $d\eta_t = \alpha_t$.

Exercise 2.8. Prove Theorem 1 for an open ball.

Exercise 2.9. Prove Theorem 1 for a union of two open balls, with the intersection diffeomorphic to an open ball.

Exercise 2.10. Prove Theorem 1 when α_t is a 1-form.

Exercise 2.11. Let $[s_t]$ be a smooth family of cohomology classes on a manifold X. Prove that it can be represented by a smooth family $s_t \in \Lambda^*(X)$.

Exercise 2.12. Let $M = U \cup V$, where U is a ball, and V a union of n balls for which Theorem 1 is already proven. Suppose that $\alpha_t = du_t$ on U and $\alpha_t = dv_t$ on V, where u_t, v_t are smooth families, $t \in [0,1]$.

a. Prove that $u_t - v_t$ is closed on $U \cap V$.

b. Suppose that $u_t - v_t$ is exact on $U \cap V$. Prove Theorem 1 in this situation.

c. Consider the Mayer-Vietoris exact sequence

$$H^{i-1}(U) \oplus H^{i-1}(V) \longrightarrow H^{i-1}(U \cap V) \xrightarrow{\delta} H^i(U \cup V) \longrightarrow H^i(U) \oplus H^i(V).$$

Let α be a closed form on $U \cup V$ such that $\alpha = du$ on U and $\alpha = dv$ on V. Let $[u - v]$ and $[\alpha]$ be cohomology classes represented by these forms. Prove that $\delta([u - v]) = [\alpha]$.

d. Prove that there exists a smooth family of cohomology classes $[a_t] \in H^{k-1}(V)$ such that $[a_t]\big|_{U \cap V} = [u_t - v_t]$.

e. Prove that the antiderivatives u_t, v_t can be chosen in such a way that $[u_t - v_t]$ is exact.

Hint: Use Exercise 2.11.

f. Prove that there exists a smooth family η_t such that $d\eta_t = \alpha_t$.

Exercise 2.13. Let A_0, A_1 be volume forms on a manifold M, with $\int_M A_0 = \int_M A_1$.

a. Prove that $A_\lambda := \lambda A_1 + (1 - \lambda) A_0$ is always a volume form. Prove that $\frac{dA_\lambda}{d\lambda} = d(i_{X_\lambda} A_\lambda)$ for a vector field X_λ smoothly depending on λ.

b. Let Ψ_λ be the exponent of the vector field X_λ defined above. Prove that $\Psi_\lambda(A_0) = A_\lambda$.

c. Let A be the set of all volume forms of constant volume on an oriented compact manifold M. Prove that the group $\operatorname{Diff}(M)$ of diffeomorphisms acts on A transitively.

Definition 2.2. Let (M, I) be an almost complex manifold, and ω a symplectic form. We say that ω is **tamed by** I if $\omega(Ix, x) > 0$ for any non-zero vector $x \in T_m M$.

Exercise 2.14. Let (M, I) be an almost complex manifold, and ω_1, ω_2 two symplectic forms tamed by I. Assume that ω_1 is cohomologous to ω_2. Prove that there exists $h \in \operatorname{Diff}(M)$ such that $h^* \omega_1 = \omega_2$.