Symplectic handout 3: Symplectic reduction

Definition 3.1. Let (M, ω) be a symplectic manifold. A vector field $v \in TM$ is called **symplectomorphic** if $\operatorname{Lie}_v \omega = 0$.

Exercise 3.1. Let ρ be a closed k-form on M, and v a vector field. Prove that $\operatorname{Lie}_v \rho = 0$ if and only if $d(i_v\rho) = 0$, where $i_v(\rho) = \rho(v, \cdot, ..., \cdot)$ denotes the contraction operator.

Definition 3.2. A symplectomorphic vector field v on (M, ω) is called **Hamiltonian** if the 1-form $i_v(\omega)$ is exact. **Hamiltonian symplectomorphism** is Ψ_1, where Ψ_t is a flow of symplectomorphisms associated with a smooth family v_t, $t \in [0,1]$ of Hamiltonian vector fields, with $\Psi_0 = \mathsf{Id}$, and $\Psi_t^{-1} \frac{d\Psi_t}{dt} = v_t$.

Definition 3.3. Let G be a Lie group acting on a symplectic manifold (M, ω) by symplectomorphisms, $\mathfrak{g} = T_e G$ its Lie algebra, and $\rho: \mathfrak{g} \longrightarrow TM$ the corresponding Lie algebra action, that is, the differential of the action of G in $e \in G$. Suppose that all vector fields in $\rho(\mathfrak{g})$ are Hamiltonian. Consider a linear map $\mu_1: \mathfrak{g} \longrightarrow C^\infty M$ which takes each vector field $x \in \mathfrak{g}$ to the Hamiltonian of $\rho(x)$, and let $\mu: M \longrightarrow \mathfrak{g}^*$ take $X \in \mathfrak{g}$ to $\mu_1(X)$. The map $\mu: M \longrightarrow \mathfrak{g}^*$ is called **the moment map** of the Lie group action. It is called **an equivariant moment map** if $\mu(g(x)) = \mathsf{coad}(g)(\mu(x))$, where $g \in G$, and coad denotes the coadjoint action of G on $\mathfrak{g}^*$.

Exercise 3.2. Let G be a Lie group acting on a simply connected symplectic manifold by symplectomorphisms. Prove that the moment map $\mu: M \longrightarrow \mathfrak{g}^*$ always exists.[1] Prove that the set of moment maps is a finite-dimensional affine space with free, transitive action of the space $\mathfrak{g}^*$ considered as an abelian Lie group.

Exercise 3.3. Let G be a Lie group which acts on M by Hamiltonian symplectomorphisms. We say that this action is **locally free** if each orbit is diffeomorphic to a quotient of G by a discrete subgroup. Prove that G acts locally freely if and only if the moment map μ is a submersion.

Exercise 3.4. Let $\mu: M \longrightarrow \mathfrak{g}^*$ be an equivariant moment map, $\rho: \mathfrak{g} \longrightarrow TM$ the Lie algebra action, and $x, y \in \mathfrak{g}$. Prove that $\omega(\rho(x), \rho(y)) = \mu([x, y])$.

Exercise 3.5. An covector $t \in \mathfrak{g}^*$ is called **central** if it is coad-invariant. Prove that any central element vanishes on $[\mathfrak{g}, \mathfrak{g}]$.

Exercise 3.6. Prove that the space of equivariant moment maps is a finite-dimensional affine space with free, transitive action of the space $(\mathfrak{g}^*)^G$ of central covectors.

Exercise 3.7. Let $\omega = \sum_i dx_i \wedge dy_i$ be the standard symplectic form on $\mathbb{C}^n$, and $X \in \mathfrak{u}(n)$ a complex linear vector field acting by isometries.

a. Prove that the symplectic gradient $\omega^{-1}(dH)$ of a function $H \in C^\infty(\mathbb{C}^n)$ can be written as $I(\mathsf{grad}\, H)$, where $\mathsf{grad}\, H = (dH)^\sharp$ is the usual gradient.

b. Let X be a vector field acting on $\mathbb{C}^n$ by linear holomorphic isometries. Prove that $4\omega = dId(r)$, where $r(z) = |z|^2$. Prove that $\operatorname{Lie}_X(Idr) = 0 = d(\langle Idr, X\rangle) + 2i_X\omega$. Deduce from this $\operatorname{Lie}_{IX}(dr) = d(\langle dr, IX\rangle) = 2i_X\omega$.

c. Prove that the moment map for the action of the Lie group $\mathbb{R} = e^{tX}$ can be written as $\mu(v) = \frac{1}{4}\operatorname{Lie}_{IX}(r)$.

[1] Here we don't assume that μ is equivariant.

Exercise 3.8. Let $\mu : M \longrightarrow \mathfrak{g}^*$ be an equivariant moment map, and $t \in \mathfrak{g}^*$ a central covector. Prove that for any and any $x, y \in \rho(g)\Big|_{T(\mu^{-1}(t))}$, one has $\omega(x, y) = 0$.

Hint. Use Exercise 3.4 and Exercise 3.5.

Exercise 3.9. Let G be a connected Lie group which acts on (M, ω) locally freely by Hamiltonian symplectomorphisms, $\mu : M \longrightarrow \mathfrak{g}^*$ an equivariant moment map, $t \in \mathfrak{g}^*$ a central element and $Z := \mu^{-1}(t)$.

a. Prove that Z is coisotropic (that is, for each $z \in Z$ one has $(T_z Z)^\perp \subset T_z Z$).

b. Prove that $(T_z Z)^\perp = \rho(\mathfrak{g})$, where $\rho : \mathfrak{g} \longrightarrow TM$ is the corresponding action of the Lie algebra $\operatorname{Lie}(G)$.

Hint. Use the previous exercise.

Exercise 3.10. Let $Z \subset M$ be a coisotropic submanifold, and $B \subset TZ$ the sub-bundle defined as $B := (TZ)^\perp$. Prove that $[B, B] \subset B$.

Remark 3.1. By Frobenius theorem, the condition $[B, B] \subset B$ means that B is is a tangent bundle to a foliation on M. This foliation is called **the characteristic foliation** of $Z \subset M$. Further on, you are allowed to apply Frobenius theorem, if you can state it correctly.

Exercise 3.11. Let $Z \subset M$ be a coisotropic subvariety in (M, ω), and $\pi : Z \longrightarrow Z_0$ the projection to the leaf space of the characteristic foliation.

a. Let $\pi : X \longrightarrow Y$ be a smooth submersion, and $\alpha \in \Lambda^k(X)$ a closed form. Prove that $\alpha = \pi^* \alpha_0$ if and only if $i_v \alpha = 0$ for any vector field v tangent to the fibers of π.

b. Prove that Z_0 is equipped with a symplectic form ω_0 such that $\pi^* \omega_0 = \omega\Big|_Z$.

Exercise 3.12. In assumptions of Exercise 3.9, let $Z := \mu^{-1}(t)$, where $t \in \mathfrak{g}^*$ is a central covector.

a. Prove that Z is G-invariant.

b. Prove that the leaves of the characteristic foliation on Z coincide with the orbits of G.

c. Assume that G is compact and its action is free. Prove that the quotient space $\frac{\mu^{-1}(t)}{G}$ is a smooth manifold which is equipped with a natural symplectic structure.

Hint. Use the previous exercise.

Definition 3.4. The manifold $M/\!\!/G := \frac{\mu^{-1}(t)}{G}$ is called **the symplectic reduction** of M.

Exercise 3.13. Consider the natural action of the group $U(n+1)$ on $\mathbb{C}P^n$.

a. Prove that there exists an $U(n+1)$-invariant non-degenerate 2-form on $\mathbb{C}P^n$.

b. Prove that such a form is always closed.

c. Prove that it is unique up to a constant multiplier.[2]

Exercise 3.14. Consider the action of $U(1)$ on $\mathbb{C}^{n+1}$ by complex rotations. Prove that the symplectic quotient $\mathbb{C}^{n+1}/\!\!/U(1)$ is $\emptyset$, a point, or $\mathbb{C}P^n$ with the Fubini-Study symplectic form, depending on the choice of the moment map.

Exercise 3.15 (*). Let $\gamma \subset S^1 \times S^1$ be a circle $S^1 \times \{x\}$, and g a Hamiltonian diffeomorphism of $S^1 \times S^1$. Prove that $g(\gamma) \cap \gamma$ is non-empty.

[2] This form is called **the Fubini-Study symplectic form**.