Symplectic handout 4: Hamiltonian isotopies

Exercise 4.1. Let X, Y be symplectomorphic vector fields on (M, ω). Prove that $[X, Y]$ is Hamiltonian.

Exercise 4.2. Let ω be an exact symplectic form, $\omega = d\eta$, and and $v := \omega^{-1}(\eta)$.

a. Prove that $\operatorname{Lie}_v \omega = \omega$.

b. Prove that $\operatorname{Lie}_v \eta = \eta$.

Exercise 4.3. Let (M, ω) be a symplectic manifold such that $\omega = d\eta$, and $X \in TM$ a vector field.

a. Prove that X is symplectomorphic if and only if only if $\operatorname{Lie}_X \eta$ is closed.

b. Prove that that X is Hamiltonian if and only if only if $\operatorname{Lie}_X \eta$ is exact.

Exercise 4.4. Let $\eta \in \Lambda^1(\mathbb{R}^{2k+1})$ be a 1-form such that the $(2k+1)$-form $\eta \wedge (d\eta)^k$ is non-degenerate, and v a vector field which satisfies $\eta(v) = 1$ and $\operatorname{Lie}_v(\eta) = 0$. Prove that v is uniquely determined by these conditions.

Exercise 4.5. Let Θ be a volume form on M. We call a vector field $v \in TM$ **Θ-hamiltonian** if $i_v\Theta$ is exact. Suppose that $v \in TM$ is Θ-Hamiltonian, and $w \in TM$ satisfies $\operatorname{Lie}_w \Theta = 0$. Prove that $[v, w]$ is Θ-Hamiltonian.

Definition 4.1. Let H_t be a family of functions on M smoothly depending on $t \in [0, 1]$, and $X_t := \omega^{-1}(dH_t)$ the corresponding Hamiltonian vector field. A **Hamiltonian isotopy** associated with H_t is a diffeomorphism flow Ψ_t which is tangent to X_t for all t, that is, satisfies $\Psi_t^{-1}\frac{d\Psi_t}{dt} = X_t$, and satisfies $\Psi_0 = \mathsf{Id}$. A **Hamiltonian symplectomorphism** is the end point of a Hamiltonian isotopy, that is, Ψ_1.

Exercise 4.6. Prove that a non-trivial rotation of a 2-torus $T^2 = S^1 \times S^1$ along one of the two circles is never a Hamiltonian symplectomorphism.

Exercise 4.7. Prove that the group of Hamiltonian symplectomorphism is a normal subgroup of the group of symplectomorphisms.

Definition 4.2. A submanifold $S \subset M$ of a symplectic manifold (M, ω) is called **Lagrangian** if $\dim S = \frac{1}{2} \dim M$ and $\omega\big|_S = 0$.

Exercise 4.8. Let $S \subset T^*M$ be a submanifold of a symplectic manifold T^*M with the standard symplectic form.

a. Suppose that $S = \Gamma_\zeta$ is a graph of a 1-form $\zeta \in \Lambda^1 M$ considered as a map $M \longrightarrow T^*M$. Prove that Γ_ζ is Lagrangian if and only if ζ is closed.

b. Let Γ_ξ be a graph of an exact 1-form ξ. Find a Hamiltonian symplectomorphism which takes the zero section Γ_0 to Γ_ξ.

c. Let M be compact. Find an example of a compact, connected Lagrangian submanifold $S \subset T^*M$ which is not obtained as a graph of a closed 1-form.

Exercise 4.9. Let $U(1)^{n+1} \subset U(n+1)$ be the subgroup of rotations which are written diagonally in a certain basis in $\mathbb{C}^{n+1}$, acting on $\mathbb{C}P^n$, and $\mu : \mathbb{C}P^n \longrightarrow (u(1)^{n+1})^*$ the corresponding moment map.

a. Prove that all fibers of μ are Lagrangian subvarieties in $\mathbb{C}P^n$. Prove that all smooth fibers are tori.

b. Prove that $\mu(\mathbb{C}P^n)$ vanishes on the diagonal $\sqrt{-1}\,\mathsf{Id}_{\mathbb{C}^n} \subset u(1)^{n+1}$

Exercise 4.10. Let ω be a non-degenerate 2-form, and $v_1, ..., v_{2n}$ a collection of commuting vector fields such that $\operatorname{Lie}_{v_i} \omega = 0$ for all i. Suppose that at some point $x \in M$ the vectors $v_i\big|_x$ define a basis in T_xM. Prove that ω is closed in a certain neighbourhood of x.

Hint. Find a coordinate system $z_1, ..., z_{2n}$ such that $v_i = \frac{d}{dz_i}$ and express ω in these coordinates.

Exercise 4.11 (*). Prove Exercise 4.10 without assuming that v_i commute, or find a counterexample.

Exercise 4.12 (*). Find a non-degenerate 2-form ω on $\mathbb{R}^6$ such that the Lie algebra of vector fields v such that $\operatorname{Lie}_v \omega = 0$ is finite-dimensional.