Symplectic handout 5: Hausdorff metric on convex sets

Definition 5.1. Let M be a metric space, and $X \subset M$ a subset. The **ε-neighbourhood** of X is $X(\varepsilon) := \bigcup_{x\in X} B_\varepsilon(x)$, where $B_\varepsilon(x)$ is an ε-ball centered in $x \in X$. **the Hausdorff distance** $d_H(X,Y)$ is the infimum of all ε such that $X(\varepsilon) \supset Y$ and $Y(\varepsilon) \supset X$.

Definition 5.2. **Diameter** diam of a metric space M is supremum $d(x,y)$ for all $x, y \in M$.

Exercise 5.1. Let M be a metric space of finite diameter, and $\mathfrak{U}$ the set of all closed subsets of M. Prove that d_H defines a metric on $\mathfrak{U}$.

Exercise 5.2. Let X_i be a d_H-Cauchy sequence of closed subsets of a metric space M, and X its limit.

a. Prove that $X = \bigcap X_i(\varepsilon_i)$, for an appropriate sequence $\{\varepsilon_i \in \mathbb{R}^{>0}\}$ converging to 0.

b. Let $\{x_i \in X_i\}$ be any sequence, and x its limit. Prove that $x \in X$.

c. Let $x \in X$. Prove that x is a limit of a sequence $\{x_i \in X_i\}$.

d. Prove that X is the set of all limiting points $\lim_{i\to\infty} x_i$, for al sequences $\{x_i \in X_i\}$

e. Prove that the topology on $\mathfrak{U}$ induced from d_H is determined by the topology on M.

Exercise 5.3. Let M be a complete metric space of finite diameter, and $\mathfrak{U}$ the set of all closed subsets of M, equipped with d_H-topology. Prove that $\mathfrak{U}$ is complete.

Exercise 5.4. Let M be a compact metric space, and $\mathfrak{U}$ the set of all closed subsets of M, equipped with d_H-topology. Prove that $\mathfrak{U}$ is compact.

Definition 5.3. Let X, Y be metric spaces. **Uniform topology** on the space $\operatorname{Map}(X,Y)$ of continuous maps is the topology induced by the metric

$$d(f,g) := \sup_{x\in X} d(f(x), g(x)).$$

C^0-topology is uniform convergence on compacts, and C^i-topology is uniform convergence on compacts with all derivatives up to i-th.

Exercise 5.5. Let X, Y be compact metric spaces, and $f, g \in \operatorname{Map}(X,Y)$. Define $d_H(f,g)$ as the Hausdorff distance between the graphs of f and g. Prove that the topology induced by this metric on $\operatorname{Map}(X,Y)$ coincides with the uniform topology.

Definition 5.4. A **convex hull** of $U \subset \mathbb{R}^n$ is the smallest convex set containing U. **A simplex** is a convex hull of $n+1$ points.

Exercise 5.6. Let $\hat{U} \subset \mathbb{R}^n$ be a convex hull of $U \subset \mathbb{R}^n$. Prove that $\hat{U}$ is a union of all simplices with vertices in U.

Exercise 5.7. Let $A \subset \mathbb{R}^n$ be a compact, convex subset of $\mathbb{R}^n$. A point $x \in A$ is called **extremal** if $x \neq ty + (1-t)z$ for any $t \in]0,1[$ and any $y, z \in A$. Prove that A is a convex hull of the set of its extremal points.

Remark 5.1. From now on, we consider $\mathbb{R}^n$ as a metric space with the standard Euclidean metric.

Exercise 5.8. Let $\mathfrak{C}$ be the set of all convex, compact subsets of $\mathbb{R}^n$, Prove that $(\mathfrak{C}, d_H)$ is complete.

Exercise 5.9. Let $\mathfrak{O}$ be the set of all open, bounded convex subsets, and let $d_U(A, B) := d_H(\mathbb{R}^n \backslash A, \mathbb{R}^n \backslash B)$.

a. Prove that d_U is a metric, and induces the same topology on $\mathfrak{O}$ as d_H.

b. Prove that $d_U(A, B) \leqslant d_H(A, B)$. Find an example when $d_U(A, B) < d_H(A, B)$.

Exercise 5.10. For any subset $A \subset \mathbb{R}^n$, consider **the boundary** $\partial A := \bar{A} \backslash A^\circ$, where $\bar{A}$ is the closure and A° is the set of interior points of A. Find compact subsets $A, B \subset \mathbb{R}^n$ such that

a. $d_H(\partial A, \partial B) > d_H(A, B)$

b. $d_H(\partial A, \partial B) < d_H(A, B)$.

c. Prove that $d_H(\partial A, \partial B) = d_H(A, B)$ when A, B are convex.

Exercise 5.11. Let $A \subset \mathbb{R}^n$ be an open, convex, bounded subset containing 0. Denote by λA the set $\{x \in \mathbb{R}^n \mid \lambda^{-1} x \in A\}$. Prove that for each $\lambda \neq 1$ there exists ε such that $\partial A(\varepsilon) \cap \partial(\lambda A) = \emptyset$.

Exercise 5.12. Let $A \subset \mathbb{R}^n$ be an open, convex, bounded subset containing 0. Prove that for each $\delta > 0$ there exist $\varepsilon > 0$ such that for any convex B with $d_H(B, A) \leqslant \varepsilon$, one has $(1 - \delta)A \subset B \subset (1 + \delta)A$.

Hint. Use the previous exercise.

Exercise 5.13. Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth diffeomorphism, $\phi(0) = 0$. Prove that there exists $\varepsilon > 0$ such that ϕ maps an open ball $B_\delta(0)$ to a convex set, for any $\delta < \varepsilon$.

Exercise 5.14. Let $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sequence of diffeomorphisms which converges in C^2-topology to a diffeomorphism ϕ. Suppose that all ϕ_i and ϕ map 0 to 0. Prove that for a sufficiently small Euclidean open ball $E \ni 0$, almost all images $\phi_i(E)$ are convex.

Definition 5.5. Let $\mathfrak{O}$ be the set of all open, bounded convex subsets. A function $c : \mathfrak{O} \longrightarrow \mathbb{R}^{\geqslant 0}$ is called **convex capacity** if it is invariant under isometries and satisfies $c(\lambda E) = \lambda^2 c(E)$ and $c(E_1) \geqslant c(E_2)$ whenever $E_1 \supset E_2$.

Exercise 5.15. Prove that any convex capacity is continuous on $\mathfrak{O}$ in the topology defined by the Hausdorff metric.

Hint. Use Exercise 5.12.

Exercise 5.16. Let $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sequence of diffeomorphisms which converge in uniform topology to a diffeomorphism ϕ. Prove that for any compact set $E \subset \mathbb{R}^n$, $\phi_i(E)$ converges to $\phi(E)$ in the topology given by the Hausdorff metric.

Exercise 5.17. Let $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sequence of diffeomorphisms which converge in uniform topology to a diffeomorphism ϕ, and c a convex capacity. Suppose that ϕ_i and ϕ map a given convex subset $E \in \mathfrak{O}$ to a convex subset. Prove that $\lim_i c(\phi_i(E)) = c(\phi(E))$.

Exercise 5.18. Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a diffeomorphism, and $c : \mathfrak{O} \longrightarrow \mathbb{R}^{>0}$ a convex capacity. Suppose that $c(\phi(E)) = c(E)$ for any $E \in \mathfrak{O}$ such that $\phi(E)$ is convex. Let $h_\lambda(x) = \lambda x$, and let $\phi_\lambda(z) := h_\lambda(\phi(h_\lambda^{-1}(z)))$.

a. Prove that ϕ_λ converges to the differential $d\phi$ uniformly on compacts as λ goes to ∞.

b. Prove that $c(d\phi(E)) = c(E)$ for any ellipsoid $E \subset \mathbb{R}^n$.