Symplectic handout 6: Kähler reduction

We freely use the definitions given in assignment 3 ("Symplectic reduction"). A **complex manifold** is a manifold equipped with an almost complex structure I which satisfies $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$, where $T^{1,0}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ is the eigenspace of I with the eigenvalue $\sqrt{-1}$. An almost complex structure that satisfies $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ is called **integrable**.

Definition 6.1. Recall that a Riemannian metric g on an almost complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. Denote by ω the **Hermitian form** $\omega(x, y) := g(Ix.y)$; it is easy to see that ω is anti-symmetric. A Hermitian metric on a complex manifold (M, I) is called **Kähler** if $d\omega = 0$. A complex manifold equipped with a Kähler metric is called **a Kähler manifold**.

Exercise 6.1. Prove that the Fubini-Study form ω_{FS} constructed in Assignment 3 is closed, and defines a Kähler structure on $\mathbb{C}P^n$.

Definition 6.2. A foliation on a manifold M is a sub-bundle $B \subset TM$ such that $[B, B] \subset B$. By Frobenius theorem, this is equivalent to a local decomposition $U = S \times R$, of any sufficiently small open set $U \subset M$, with $B \subset TU$ equal to the tangent bundle to the fibers of the projection $U \longrightarrow R$. **A leaf** of a foliation is a maximal connected immersed submanifold $Z \longrightarrow M$ which satisfies $T_z Z = B\big|_z$ at each $z \in Z$. **Projection to the leaf space** is a smooth submersion $U \longrightarrow R$, defined locally in U, and mapping U to the set of leaves of B on U.

Definition 6.3. Transversal Riemannian structure/symplectic structure/almost complex structure on a foliated manifold $(M, B \subset TM)$ is a scalar product/skew-symmetric form/almost complex structure on TM/B which is locally obtained as a pullback of a Riemannian/symplectic/almost complex structure on the leaf space.

Exercise 6.2. Let G be a compact Lie group acting on a manifold M. Prove that all orbits have dimension $\dim G$ if and only if for some basis $g_1, ... g_n \in \operatorname{Lie} G$ the corresponding vector fields on M are linearly independent everywhere.

Definition 6.4. In this case, the action of G on M is called **locally free.**

Exercise 6.3. Let M be a manifold equipped with a locally free action of a compact Lie group G and $B \subset TM$ the bundle of vectors tangent to the G-action. Suppose that TM/B is equipped with a G-invariant metric and almost complex structure. Prove that these structures are transversal, in the sense of the Definition 6.3, and the orbit space is almost complex Hermitian.

Exercise 6.4. Let M be an almost Kähler manifold, G a compact Lie group acting on M locally freely by Hamiltonian isometries, $t \in \mathfrak{g}^*$ a central element, and $\mu : M \longrightarrow \mathfrak{g}^*$ an equivariant moment map. Denote by $K \subset T\mu^{-1}(t)$ the bundle of vectors tangent to the orbits of G on $\mu^{-1}(t)$.

a. Prove that $K \subset T\mu^{-1}(t)$ is a foliation, and $(\mu^{-1}(t), K \subset T\mu^{-1}(t))$ is equipped with a transversal Riemannian and a transversal symplectic structure, obtained by restricting the Riemannian and symplectic forms to $K^\perp$.

b. Prove that these 2-forms define an almost Kähler structure on the orbit space.[1]

[1] **An almost Kähler structure** is a pair of compatible almost complex and symplectic structures.

Hint. Use the arguments from Assignment 3.

Exercise 6.5. Let M be an almost Kähler manifold, G a compact Lie group acting on M locally free by Hamiltonian isometries, $t \in \mathfrak{g}^*$ a central element, $\mu: M \longrightarrow \mathfrak{g}^*$ an equivariant moment map. Denote by $K \subset T\mu^{-1}(t)$ the bundle of vectors tangent to the orbits of G.

a. Prove that $K \cap I(K) = 0$.

b. Prove that $T\mu^{-1}(t)/K = TM\Big|_{\mu^{-1}(t)}/K_{\mathbb{C}}$, where $K_{\mathbb{C}} = K \oplus I(K)$. Prove that the complex structure operator on $T\mu^{-1}(t)/K$ defined in Exercise 6.4 coincides with the complex structure on $TM\Big|_{\mu^{-1}(t)}/K_{\mathbb{C}}$ induced by the almost complex structure $I \in \operatorname{End}(TM)$.

Exercise 6.6. Let G be a compact Lie group freely acting on a manifold M. Consider the orbit space M/G with the quotient topology. Prove that M/G is homeomorphic to a smooth manifold.

Exercise 6.7. Let M be a manifold equipped with a locally free action on compact Lie group G. Prove that M/G is locally homeomorphic to $\mathbb{R}^n/G$, where G is a finite group.

Exercise 6.8. Construct a smooth manifold M equipped with a free action of S^1 such that M/S^1 is a 2-torus, and a $\mathbb{Z}/2\mathbb{Z}$-quotient M_1 of M with a locally free action of S^1 such that M_1/S^1 is homeomorphic to S^2.

Definition 6.5. Let $B \subset TM$ be a sub-bundle equipped with a complex structure operator $I \in \operatorname{End} B$, $I^2 = -\mathsf{Id}$, and $B \otimes_{\mathbb{R}} \mathbb{C} = B^{1,0} \oplus B^{0,1}$ the corresponding eigenspace decomposition, $I\Big|_{B^{1,0}} = \sqrt{-1}$, $I\Big|_{B^{0,1}} = -\sqrt{-1}$. The pair (B, I) is called **a CR-structure on** M if $[B^{1,0}, B^{1,0}] \subset B^{1,0}$.

Exercise 6.9. Let $Z \subset M$ be a submanifold of a complex manifold. Assume that $B := TZ \cap I(TZ)$ has constant rank. Prove that $(B, I\Big|_B)$ is a CR-structure on M.

Exercise 6.10. Let (Z, B, I) be a CR-manifold, and G a compact Lie group freely acting on Z. Assume that $TZ = B \oplus K$, where $K \subset TZ$ is the subspace generated by the vector fields tangent to the G-action. Suppose that the operator I on B defines a transversal complex structure with respect to the foliation tangent to K. Prove that the natural almost complex structure on Z/G induced by the action of I on $B = TZ/K = T(Z/G)$ is integrable.

Exercise 6.11. Let M be a Kähler manifold, G a compact Lie group freely acting on M by Hamiltonian isometries, $t \in \mathfrak{g}^*$ a central element, and $\mu: M \longrightarrow \mathfrak{g}^*$ an equivariant moment map. Prove that the quotient $M/\!\!/G := \frac{\mu^{-1}(t)}{G}$ is equipped with a natural Kähler structure.

Hint. Use Exercise 6.4 to show that $M/\!\!/G$ is almost Kähler, Exercise 6.9 to show that $\mu^{-1}(t)$ is a CR-manifold, and Exercise 6.10 to prove that the complex structure on $M/\!\!/G$ is integrable.

Exercise 6.12. Prove that the Kähler metric on the standard symplectic quotient $\mathbb{C}^n/\!\!/S^1 = \mathbb{C}P^{n-1}$ is proportional to the Fubini-Study metric.