Symplectic handout 7: Geometric invariant theory

We freely use the definitions given in assignment 3 and 6 ("Symplectic reduction", "Kähler reduction").

Definition 7.1. Let M be a complex manifold. Define $d^c: \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$ as $d^c := IdI^{-1}$, where I acts on k-forms multiplicatively. **Kähler potential** is a function $\phi: M \longrightarrow \mathbb{R}$ such that $dd^c\phi$ is a Kähler form.

Definition 7.2. **A holomorphic vector field** is a vector field satisfying $\operatorname{Lie}_X I = 0$, that is, such that the corresponding diffeomorphism flow e^{tX} is holomorphic.

Exercise 7.1. Let G be a compact Lie group acting on a complex manifold holomorphically and preserving a Kähler potential ϕ. Denote by $\omega := dd^c\phi$ the corresponding Kähler form.

a. Prove that G acts on (M, ω) by isometries.

b. Let $X \in \operatorname{Lie}(G)$ be a vector field on TM tangent to the action of G. Prove that $i_X\omega = i_X(dd^c\phi) = -d(i_X(d^c\phi))$.

c. Prove that the function $-\langle X, d^c\phi\rangle$ is a Hamiltonian for X.

d. Prove that the moment map for the action of G can be written as $(m, X) \longrightarrow \operatorname{Lie}_{IX}(\phi)(x)$.

Exercise 7.2. Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure. Prove that its action admits an equivariant moment map μ, given by $\langle \mu(v), g\rangle = \operatorname{Lie}_{Ig} l$, where $v \in V$, $g \in \operatorname{Lie}(G)$, and $l \in C^\infty V$ the function $l(v) = \frac{1}{4}|v|^2$.

Hint. Use the previous exercise.

Exercise 7.3. Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure, and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ its complexification. Denote by $\mu: V \longrightarrow \mathfrak{g}^*$ the moment map. Prove that a vector $z \in V$ belongs to $\mu^{-1}(0)$ if and only if the function $l: G_{\mathbb{C}} \cdot z \longrightarrow \mathbb{R}$ on the orbit $G_{\mathbb{C}}$ has extremum in z.

Exercise 7.4. Let $V = \mathbb{C}^n$, $A \in \mathfrak{u}(V)$ be an anti-Hermitian endomorphism, and $G_A := e^{tA} \subset GL(V)$, $t \in \mathbb{C}$ the corresponding 1-parametric subgroup.

a. Consider $l(e^{tA}z) := |e^{tA}z|^2$ as a function on $G_A \cdot z$. Prove that $l(e^{(t+u)A}(z)) = l(e^{tA}(z))$ for all $u \in \mathbb{R}$. Prove that $\frac{d^2}{du^2}|e^{(t+\sqrt{-1}u)A}(z)|^2 = |A(e^{tA}(z))|^2$.

b. Assume that $A(z) \neq 0$. Prove that l is convex on the complex line $G_A := e^{\mathbb{C}A}$ and has at most one minimum on the real line $e^{\sqrt{-1}\mathbb{R}A}(z)$.

c. Let A be diagonalized in an orthonormal basis $x_1, ..., x_n \in \mathbb{R}$, such that $A(x_i) = \sqrt{-1}w_i x_i$, and $z = \sum \alpha_i x_i$. Prove that the function l has a minimum on the line $e^{\sqrt{-1}\mathbb{R}A}(z)$ if and only if there are two basis vectors x_l, x_k with $\alpha_l, \alpha_k \neq 0$, such that $w_l < 0$ and $w_k > 0$.

Exercise 7.5. Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure, and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ its complexification.

a. Consider $l(z) = |z|^2$ as a function on $G_{\mathbb{C}} \cdot z$. We parametrize $G_{\mathbb{C}} \cdot z$ by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ using the map $\mathfrak{g}_{\mathbb{C}} \xrightarrow{g \mapsto e^g z} G_{\mathbb{C}} \cdot z$. Prove that $\phi(g) := l(e^g z)$ is a convex function which satisfies $\frac{d^2}{dg^2}(\phi)(g) = |\operatorname{im}(g)(z)|^2$.

b. Prove that either l has no extremal points on $G_{\mathbb{C}} \cdot z$, or l takes minimum somewhere on $G_{\mathbb{C}} \cdot z$. Prove that G acts transitively on the set of minima of l on $G_{\mathbb{C}} \cdot z$.

Definition 7.3. Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure, and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ its complexification. An orbit $G_{\mathbb{C}} \cdot z$, $z \neq 0$ is called **stable** if l reaches minimum on $G_{\mathbb{C}} \cdot z$, **unstable** if 0 belongs to the closure of $G_{\mathbb{C}} \cdot z$, and **(strictly) semistable** if if it is not stable and not unstable.

Exercise 7.6. Let $G_{\mathbb{C}} \cdot z \subset V$ be a stable orbit.

a. Prove that for any non-zero $g \in \sqrt{-1}\mathfrak{g}$, one has $\lim_{t\to\infty} |e^{tg}(z)| = \infty$.

b. Let $\bar{B}_R \subset V$ a closed ball of radius R. Prove that $\bar{B}_R \cap G_{\mathbb{C}} \cdot z$ is compact for all $R \in \mathbb{R}^{>0}$.

c. Prove that there is a neighbourhood $U \ni z$ such that for all $z_i \in U$, the orbit $G_{\mathbb{C}} \cdot z_1$ is stable.

Remark 7.1. The following theorem is identifies "the GIT reduction" (taking a $G_{\mathbb{C}}$-quotient of the union of all stable orbits) and the symplectic reduction.

Exercise 7.7. Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure, and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ its complexification. Denote by $\mu : V \longrightarrow \mathfrak{g}^*$ the moment map, $\mu(g, z) := (\operatorname{Lie}_{Ig} l)(z)$.

a. Prove that an orbit $G_{\mathbb{C}} \cdot z$ is stable if and only if $G_{\mathbb{C}} \cdot z \cap \mu^{-1}(0) \neq 0$.

b. Prove that $G_{\mathbb{C}} \cdot z \cap \mu^{-1}(0)$ is precisely one G-orbit.

c. Prove that $\mu^{-1}(0)/G = V_s/G_{\mathbb{C}}$, where $V_s \subset V$ is the union of all stable orbits.

Hint. Use Exercises 7.3 and 7.5.

Exercise 7.8. ("Hilbert-Mumford criterion of stability")
Let $G = U(1)$ act on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure, $G_{\mathbb{C}} = \mathbb{C}^*$ the corresponding complex Lie group, and $z \in V$ a non-zero vector.

a. Prove that there exists an orthonormal basis $x_1, ..., x_n$ in V such that $g(x_i) = \sqrt{-1} w_i x_i$, where $w_i \in \mathbb{Z}$ are integer numbers called **the weights** of the action.

b. Let $z = \sum \alpha_{l_k} x_{l_k}$, where α_{l_k} are all non-zero. Prove that $G_{\mathbb{C}} \cdot z$ is unstable if and only if all w_{l_k} are positive or negative.

c. Prove that $G_{\mathbb{C}} \cdot z$ is stable if and only if some w_{l_k} are positive while others are negative. Prove that it is strictly semistable if some w_{l_k} vanish and all others are positive or negative.

Exercise 7.9. Let $G = U(1)$ act on $V = \mathbb{C}^2$ as $\rho_t(z_1, z_2) = (tz_1, t^{-1}z_2)$. Find all stable and non-stable orbits, and prove that $V /\!\!/ G = \mathbb{C}^*$.