Symplectic geometry

lecture 1: Symplectic manifolds and symplectic capacities

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Symplectic manifolds

DEFINITION: A **symplectic form** is a closed, non-degenerate 2-form $\omega \in \Lambda^2(M)$.

REMARK: A $2n$-manifold equipped with a non-degenerate antisymmetric 2-form ω is oriented. Indeed, ω^n is a volume form.

DEFINITION: An almost complex structure is an operator $I: TM \longrightarrow TM$ such that $I^2 = -\operatorname{Id}$.

THEOREM: Let ω be a non-degenerate 2-form on a manifold. **Then there exists an almost complex structure I such that the form $x, y \longrightarrow \omega(x, I(y))$ is symmetric and positive definite.** Moreover, **the space of such almost complex structures is contractible.**

Proof: See the exercises. ■

COROLLARY: A manifold admits a non-degenerate antisymmetric 2-form if and only if it it admits an almost complex structure.

Existence of symplectic structures

THEOREM: (Gromov)
Let M be a non-compact manifold admitting an almost complex structure. **Then M admits a symplectic structure.**

Proof: *Eliashberg-Mishachev, Introduction to the h-principle.*

By contrast, existence of a symplectic structure **puts great restrictions on the topology of a compact almost complex manifold.**

For instance, the top power of a symplectic form is the volume form, hence **its cohomology class is non-zero.**

Moser isotopy

DEFINITION: Let (M, ω) and (M', ω') be symplectic manifolds. A diffeomorphism $\varphi : M \longrightarrow M'$ is called **a symplectomorphism** if $\varphi^* \omega' = \omega$.

REMARK: Symplectic geometry is a field of mathematics which **studies symplectic manifolds up to an isomorphism and the group of symplectomorphisms.**

THEOREM: (Moser's isotopy lemma)
Let M be a compact symplectic manifold, and ω_t, $t \in [0, 1]$ a smooth deformation of a symplectic form. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a diffeomorphism flow $V_t \in \operatorname{Diff}(M)$ mapping ω_0 to ω_t.**

Proof: Next lecture.

Darboux' theorem

THEOREM: (Darboux) Let (M, ω) be a symplectic manifold. Then in a neighbourhood of each point **there exist coordinates** $p_1, ..., p_n, q_1, ..., q_n$ **such that** $\omega = \sum_i dp_i \wedge dq_i$.

Proof: Next lecture (deduced from Moser isotopy).

REMARK: From this theorem one can deduce that **the symplectomorphism group** $\operatorname{Symp}(M)$ **is simple and infinitely transitive**, that is, acts transitively on discrete subsets $Z \subset M$ of the same cardinality. The same is true for the diffeomorphism group, and the proof is more or less the same.

DEFINITION: Darboux coordinates on a symplectic manifold is a coordinate system $p_1, ..., p_n, q_1, ..., q_n$ such that $\omega = \sum_i dp_i \wedge dq_i$

Volume and capacity

DEFINITION: Standard symplectic structure on $\mathbb{R}^{2n}$ with coordinates $p_1, ..., p_n, q_1, ..., q_n$ is $\omega := \sum_i dp_i \wedge dq_i$ ("Darboux coordinates").

DEFINITION: A symplectic ball of radius r is the standard ball

$$B^{2n} := \left\{ (p_1, ..., p_n, q_1, ..., q_n) \in \mathbb{R}^{2n} \ \middle| \ \sum_i p_i^2 + q_i^2 < r^2 \right\}$$

equipped with the standard symplectic form $\omega := \sum_i dp_i \wedge dq_i$

DEFINITION: Symplectic volume of a symplectic manifold (M, ω), $\dim_{\mathbb{R}} M = 2n$, is $\operatorname{Vol}(M, \omega) := \int_M \omega^n$.

Symplectic capacity **is a way to distinguish diffeomorphic symplectic manifolds of the same volume.**

Volume and capacity (2)

EXERCISE: Let ν be a volume form on a compact manifold M, ν' a volume form on M', diffeomorphic to M. Assume that $\int_M \nu = \int_{M'} \nu'$. **Prove that there exists a diffeomorphism $\varphi : M \longrightarrow M'$ such that $\varphi^* \omega' = \omega$.**

EXERCISE: Let ν be a volume form on a manifold M, not necessarily compact, and ν' a volume form on M', diffeomorphic to M. Assume that $\int_M \nu = \int_{M'} \nu'$. **Prove that for any open subset $U \subset M$ with compact closure there exists a diffeomorphism $\varphi : M \longrightarrow M'$ such that $\varphi^* \omega' |_U = \omega|_U$.**

QUESTION: Let $A, B \subset \mathbb{R}^{2n}$ be two manifolds homeomorphic to a ball, of the same symplectic volume. **Are they symplectomorphic?**

ANSWER: Not always, because **the symplectic capacity might be different.**

Gromov capacity

DEFINITION: An (open) symplectic embedding is an open embedding of symplectic manifolds, symplectimorphic to its image.

DEFINITION: Let (M, ω) be a symplectic manifold, and r a supremum of radii of all symplectic balls of the same dimension, admitting a symplectic embedding to M. The number $\operatorname{capa}(M, \omega) := \pi r^2$ is called **Gromov symplectic capacity** of M.

THEOREM: (Ekeland-Hofer)
Let φ be an oriented diffeomorphism of symplectic manifold. Then **φ is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.**

Proof: Later in these lectures. ∎

Gromov Non-Squeezing Theorem

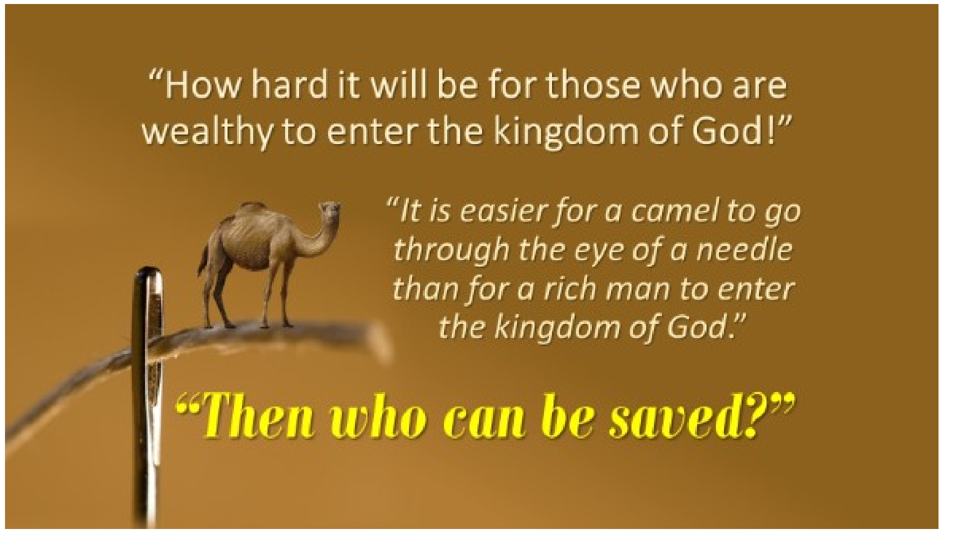
DEFINITION: A symplectic cylinder C_r is $\mathbb{R}^{2n} \times B_r$, where $\mathbb{R}^{2n}$ is equipped with the standard symplectic form $\sum_i dp_i \wedge dq_i$, and B_r is the standard symplectic ball of radius r in $\mathbb{R}^2$.

THEOREM: (Gromov) Symplectic capacity of a symplectic cylinder C_r is equal to πr^2.

Proof: Later in these lectures. ■

REMARK: The volume is not the only obstruction to symplectic embeddings. Indeed, **the volume of the symplectic cylinder is infinite.**

Related: This theorem is also called "Symplectic camel theorem", or "Gromov Non-Squeezing Theorem".



Symplectic packing

EXERCISE: Prove that **there exists a non-degenerate $U(n+1)$-invariant symplectic form ω on $\mathbb{C}P^n$.** Prove that such ω is unique, up to a constant multiplier. Prove that **this form is closed.**

DEFINITION: The form ω is called **the Fubini-Study form**. We usually fix the constant multiplier in such a way that Vol$(\mathbb{C}P^n, \omega) = 1$.

THEOREM: Let v_N be a supremum of the total volume of N equal symplectic balls which admit a disjoint symplectic embedding to $\mathbb{C}P^2$ of volume 1. Then

N	1	2	3	4	5	6	7	8	9	$N > 9$
ν_N	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1	1

The first 4 numbers are due to Gromov, last to Biran, the rest are McDuff-Polterovich.

Nagata conjecture

REMARK: These numbers are related to Nagata conjecture, which is still unsolved (Biran used Taubes' work on Seiberg-Witten invariants to avoid proving it).

CONJECTURE: Suppose $p_1, ..., p_r$ are very general points in $\mathbb{C}P^2$ and that $m_1, ..., m_r$ are positive integers. **Then for any $r > 9$, any complex curve C in $\mathbb{C}P^2$ that passes through each of the points p_i with multiplicity m_i must satisfy** $\deg C > \frac{1}{\sqrt{r}} \sum_{i=1}^r m_i$.

REMARK: Nagata conjecture was known already to Nagata when r is a full square, **and unknown for all other r.** Biran's theorem is considered as the **symplectic version of Nagata's conjecture.**

An exercise for Wednesday

EXERCISE: Let $\alpha_t \in \Lambda^k(M)$, $t \in [0,1]$ be a smooth family of exact forms. Prove that **there exists a smooth family of forms $\eta_t \in \Lambda^{k-1}(M)$, $t \in [0,1]$, such that $d\eta_t = \alpha_t$.**