Symplectic geometry

lecture 2: Moser isotopy Lemma

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HSE, room 306, 16:20,

September 08, 2021

Superalgebras

DEFINITION: Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

EXAMPLE: Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

DEFINITION: Let A^* be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ be a map which shifts grading by i. It is called a **graded derivation**, if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

REMARK: If i is even, graded derivation is a usual derivation. If it is even, it an odd derivation.

DEFINITION: Let M be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **contraction with a vector field** $i_X: \Lambda^i M \longrightarrow \Lambda^{i-1} M$, mapping an i-form α to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

EXERCISE: Prove that i_X is an odd derivation.

Supercommutator

DEFINITION: Let A^* be a graded vector space, and $E: \ A^* \longrightarrow A^{*+i}$, $F: \ A^* \longrightarrow A^{*+j}$ operators shifting the grading by i, j. Define **the supercommutator** $\{E, F\} := EF - (-1)^{ij} FE$.

DEFINITION: An endomorphism of a graded vector space which shifts grading by i is called **even** if i is even, and **odd** otherwise.

EXERCISE: Prove that the supercommutator satisfies **graded Jacobi identity,**

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if E, F are even, and 1 otherwise.

REMARK: There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. **Each time when in commutative case two letters E, F are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.**

EXERCISE: Prove that a supercommutator of superderivations is again a superderivation.

Lie derivative

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along v** if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra.

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v \restriction_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

Cartan's formula

EXERCISE: Prove that $\{d, \{d, E\}\} = 0$ **for each** $E \in \operatorname{End}(\Lambda^* M)$**.**

THEOREM: (Cartan's formula) Let i_v be a convolution with a vector field, $i_v(\eta) = \eta(v, \cdot, \cdot, ..., \cdot)$ **Then** $\{d, i_v\}$ **is equal to the Lie derivative along** v**.**

Proof: $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two graded derivations is a graded derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df \rangle$. ■

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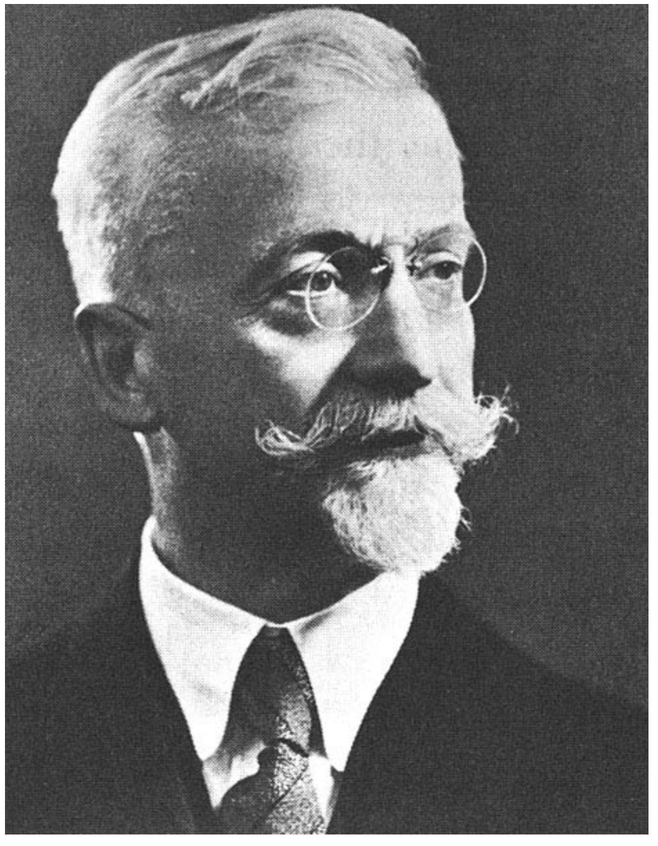
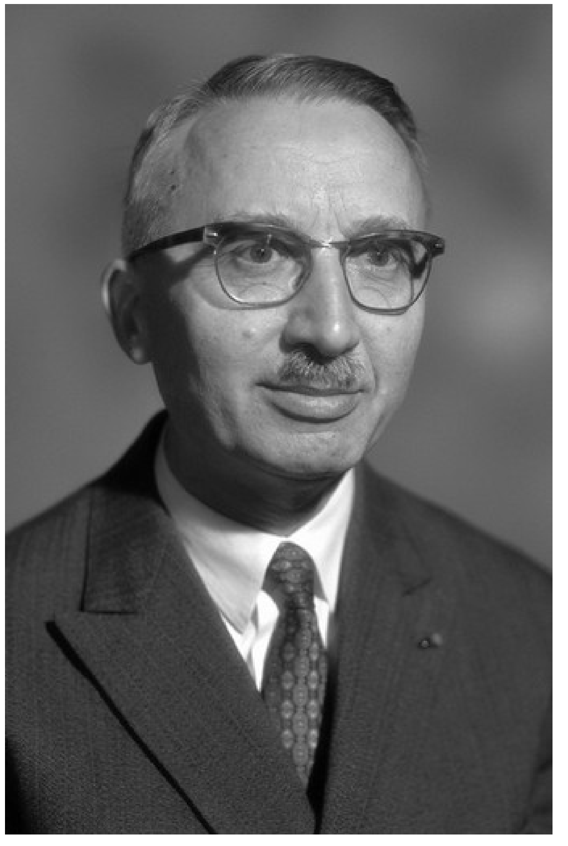
Cartan's magic formula: $d \circ i_x + i_x \circ d = \operatorname{Lie}_x$.

Which Cartan?

Élie Cartan (1869-1951) Henri Cartan (1904-2008)





Élie Cartan?
(Robert Bryant and Dick Palais, Mathoverflow)

Henri Cartan?
(S.S. Chern: Lectures on differential geometry)

Flow of diffeomorphisms

DEFINITION: Let $f: M \times [a,b] \longrightarrow M$ be a smooth map such that for all $t \in [a,b]$ the restriction $f_t := f\Big|_{M\times\{t\}}: M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $f \longrightarrow (V_c^{-1})^* \frac{dV_t}{dt}|_{t=c} f$. **Then $f \longrightarrow (V_t^{-1})^* \frac{d}{dt} V_t^* f$ is a derivation of $C^\infty M$** (that is, a vector field).

Poincaré lemma

DEFINITION: An open subset $U \subset \mathbb{R}^n$ is called **starlike** if for any $x \in U$ the interval $[0, x]$ belongs to U.

THEOREM: (Poicaré lemma) Let $U \subset \mathbb{R}^n$ be a starlike subset. **Then $H^i(U) = 0$ for $i > 0$.**

REMARK: The proof would follow if we construct a vector field $\vec{r}$ such that $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^*(M)$: $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$. Indeed, for any closed form α we would have $\alpha = \operatorname{Lie}_{\vec{r}} R\alpha = di_{\vec{r}} R\alpha + i_{\vec{r}} R d\alpha = di_{\vec{r}} R\alpha$, hence any closed form is exact.

Then Poincaré lemma is implied by the following statement.

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Radial vector field on starlike sets

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Proof. Step 1: Let t be the coordinate function on a real line, $f(t) \in C^\infty \mathbb{R}$ a smooth function, and $v := t\frac{d}{dt}$ a vector field. Define $R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda$. Then this integral converges whenever $f(0) = 0$, and satisfies $\operatorname{Lie}_v R(f) = f$. Indeed,

$$\int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda = \int_0^t \frac{f(\lambda t)}{t\lambda} d(t\lambda) = \int_0^t \frac{f(z)}{z} dz,$$

hence $\operatorname{Lie}_v R(f) = t\frac{f(t)}{t} = f(t)$.

Step 2: Consider a function $f \in C^\infty \mathbb{R}^n$ satisfying $f(0) = 0$, and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. **Then**

$$R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda$$

converges, and satisfies $\operatorname{Lie}_{\vec{r}} R(f) = f$.

Radial vector field on starlike sets (2)

Step 3: Consider a differential form $\alpha \in \Lambda^i$, and let $h_\lambda x \longrightarrow \lambda x$ be the homothety with coefficient $\lambda \in [0,1]$. Define

$$R(\alpha) := \int_0^1 \lambda^{-1} h_\lambda^*(\alpha) d\lambda.$$

Since $h_\lambda^*(\alpha) = 0$ for $\lambda = 0$, this integral converges. **It remains to prove that $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$.**

Step 4: Let α be a coordinate monomial, $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$. Clearly, $\operatorname{Lie}_{\vec{r}}(T^{-1}\alpha) = 0$, where $T = t_{i_1} t_{i_2} ... t_{i_k}$. **Since $h_\lambda^*(f\alpha) = h_\lambda^*(Tf)T^{-1}\alpha$, we have $R(f\alpha) = R(Tf)T^{-1}\alpha$ for any function $f \in C^\infty M$.** This gives

$$\operatorname{Lie}_{\vec{r}} R(f\alpha) = \operatorname{Lie}_{\vec{r}} R(Tf)T^{-1}\alpha = TfT^{-1}\alpha = f\alpha.$$

■

A smooth choice of anti-differential

THEOREM 1: Let $\alpha_t \in \Lambda^k(M)$, $t \in [0,1]$ be a smooth family of exact forms. Then **there exists a smooth family of forms $\eta_t \in \Lambda^{k-1}(M)$, $t \in [0,1]$, such that $d\eta_t = \alpha_t$.**

Proof. Step 1: Let $\{U_i\}$ be a covering of M by open balls, with all successive intersection of balls diffeomorphic to balls or empty. Such a covering can be obtained, for example, using Voronoy partitions, or a triangulation.

We use induction by the number of open balls. When there is only one ball, the statement follows from the explicit proof of Poincaré lemma. Suppose that we proved the theorem for a union of n open balls.

Let $M = U \cup V$, where U is a ball, and V a union of n balls for which the theorem is already proven. Then $\alpha_t = du_t$ on U and $\alpha_t = dv_t$ on V. The form $u_t - v_t$ is closed. **Suppose it is exact.** Since $U \cap V$ is a union of n balls, the theorem is true for it, and $u_t - v_t = dw_t$. Extending w_t to U and replacing u_t by $u'_t := u_t - dw_t$, we obtain forms u'_t, v_t which agree on $U \cap V$, and can be glued into η_t such that $d\eta_t = \alpha_t$.

A smooth choice of anti-differential (2)

Step 1, remainder: Let $M = U \cup V$, where U is a ball, and V a union of n balls for which the theorem is already proven. Then $\alpha_t = du_t$ on U and $\alpha_t = dv_t$ on V. **The form $u_t - v_t$ is closed. If it is exact, we are done.**

Step 2: It remains to prove that $u_t - v_t$ can be chosen exact. Consider the Mayer-Vietoris exact sequence

$$H^{i-1}(U) \oplus H^{i-1}(V) \longrightarrow H^{i-1}(U \cap V) \xrightarrow{\delta} H^i(U \cup V) \longrightarrow H^i(U) \oplus H^i(V).$$

The cohomology class of $u_t - v_t$ is by construction mapped to the cohomology class of α_t under the coboundary map δ. **Since α_t is exact, this class comes from $H^{i-1}(U) \oplus H^{i-1}(V)$.**

Denote the corresponding family of cohomology classes by $[s_t] := [u_t - v_t]$. Choosing a basis $x_1, ..., x_n$ in cohomology of $U \cap V$ and representing each x_i by a smooth form, we may represent $[s_t]$ by a closed form s_t which smoothly depends on t. Since $[s_t]$ comes from $H^{i-1}(U) \oplus H^{i-1}(V)$, there are families of closed forms u''_t on U and v''_t on V such that s_t is cohomologous to $u''_t - v''_t$. **Replacing u_t by $u_t - u''_t$ and v_t by $v_t - v''_t$, we obtain a new family u'''_t, v'''_t such that $u'''_t - v'''_t = u_t - v_t - s_t$ is exact.** ■

Moser isotopy lemma

THEOREM: (Moser's isotopy lemma)
Let M be a compact symplectic manifold, and ω_t, $t \in [0,1]$ a smooth deformation of a symplectic form. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a diffeomorphism flow $\Psi_t \in \operatorname{Diff}(M)$ mapping ω_t to ω_0, for all t.**

Proof. Step 1: Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. Then $\frac{d\omega_t}{dt} = d\eta_t$, where $\eta_t \in \Lambda^1(M)$. **By Theorem 1, this form can be chosen smoothly in t.**

Step 2: Let v_t be the tangent vector field to Ψ_t, with $v_t := \Psi_t^{-1}\frac{d\Psi_t}{dt}$. The equation $\Psi_t^*\omega_t = \omega_0$ (for all $t \in [0,1]$) is equivalent to $\Psi_0 = \operatorname{Id}$, $\frac{d\Psi_t}{dt}\omega_t = -\Psi_t\frac{d\omega_t}{dt}$, which is the same as

$$\operatorname{Lie}_{v_t}\omega_t = -\frac{d\omega_t}{dt}. \quad (*)$$

By Cartan's formula, $\operatorname{Lie}_{v_t}\omega_t = d(i_{v_t}(\omega_t))$. **Then (*) is equivalent to $d(i_{v_t}\omega_t) = -d\eta_t$.**

Step 3: Since ω_t is non-degenerate, there exists a unique $v_t \in TM$ such that $i_{v_t}\omega_t = -\eta_t$. Integrating the time-dependent vector field v_t to a flow of diffeomorphisms, **we obtain Ψ_t satisfying $\Psi_t^*\omega_t = \omega_0$.** ■

Hodge theory on finite-dimensional spaces

Let

$$\ldots \xrightarrow{d} C_{-1} \xrightarrow{d} C_0 \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \ldots$$

be a complex of finite-dimensional vector spaces. Put a scalar product g on each C_i, and let $d^* : C_i \longrightarrow C_{i-1}$ be adjoint operators. Since $g(dx, y) = d(x, d^* y)$, the orthogonal complement to $\operatorname{im} d$ is $\ker d^*$, and orthogonal complement to $\ker d$ is $\operatorname{im} d^*$.

Let $\Delta := dd^* + d^*d$ be **the Laplacian operator**. Then $(\Delta x, y) = (dx, dy) + (d^*x, d^*y)$, hence $x \in \ker\Delta \Leftrightarrow dx = d^*x = 0 \Leftrightarrow x \in (\operatorname{im} d)^\perp \cap (\operatorname{im} d^*)^\perp$. **This gives a direct sum decomposition $C_i = \ker\Delta \oplus \operatorname{im} d \oplus \operatorname{im} d^*$.** Since $\operatorname{im} d^* = (\ker d)^\perp$, this also gives $\ker\Delta \oplus \operatorname{im} d = \ker d$, and **identifies $\ker\Delta$ with cohomology of d.**

Since $\operatorname{im}\Delta \perp \ker\Delta$, **the operator Δ is invertible on $\operatorname{im}\Delta$.** Denote by G_Δ the operator which acts as zero on $\ker\Delta$ and as Δ^{-1} on $\operatorname{im}\Delta$. This operator is called **the Green operator**.

Write $G_d := d^* G_\Delta$. For any vector $\alpha \in \operatorname{im} d$, one has $\alpha = G_\Delta \Delta(\alpha)$ becase $\alpha \perp \ker\Delta$, **which gives $\alpha = G_\Delta dd^*\alpha = dG_\Delta\alpha$.**

A smooth choice of anti-differential (2)

THEOREM: Let $\alpha_t \in \Lambda^k(M)$, $t \in [0,1]$ be a smooth family of exact forms. Then **there exists a smooth family of forms $\eta_t \in \Lambda^{k-1}(M)$, $t \in [0,1]$, such that $d\eta_t = \alpha_t$.**

A proof using Hodge theory (works only when M is compact).
Step 1: The decomposition $C_* = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} d^*$ is always valid in finite-dimensional situation. For infinite-dimensional vector spaces, it works if and only if the spaces $\operatorname{im} d$ and $\operatorname{im} d^*$ are closed. **Hodge theory** claims that this is true, and also defines **the Green operator** G_Δ which inverts Δ on $\operatorname{im} \Delta = \ker \Delta^\perp$. Since Δ commutes with d, d^*, the same is true for G_Δ.

Step 2: Write $G_d := d^* G_\Delta$. For any exact form α, one has $\alpha = G_\Delta \Delta(\alpha)$ because $\alpha \perp \ker \Delta$, which gives $\alpha = G_\Delta d d^* \alpha = d G_\Delta \alpha$. Writing $\eta_t = G_\Delta \alpha_t$, **we obtain a smooth family η_t such that $d\eta_t = \alpha_t$.** ■

Fine resolutions

DEFINITION: Recall that a sheaf $\mathcal{F}$ over a manifold M is called **fine** if for every covering $\{U_i\}$ of M admitting a partition of unity, and any section $f \in \mathcal{F}(M)$, there exists compactly supported sections $f_i \in \mathcal{F}(U_i)$ such that $\sum_i f_i = f$.

EXAMPLE: Any sheaf of $C^\infty M$-modules is clearly fine.

REMARK: Fine sheaves are clearly **acyclic**, in other words, for any fine resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow ...$ of a sheaf $\mathcal{F}$, one has $H^i(\mathcal{F}) = H^i(H^0(\mathcal{F}_*))$ where $H^i(H^0(\mathcal{F}_*))$ denotes the cohomology of the complex of global sections $0 \longrightarrow H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{F}_1) \longrightarrow H^0(\mathcal{F}_2) \longrightarrow ...$

Fine resolutions and fiberwise closed differential forms

REMARK: Let $\mathbb{R}^\sigma_{M\times[0,1]}$ the sheaf of smooth functions which are constant along the fibers of the projection $\sigma: M\times[0,1]\longrightarrow[0,1]$. The sheaf $\mathbb{R}^\sigma_{M\times[0,1]}$ **admits a fine resolution**

$$0\longrightarrow \mathbb{R}^\sigma_{M\times[0,1]}\longrightarrow C^\infty(M\times[0,1])\xrightarrow{d_\sigma}\Lambda^1_\sigma(M\times[0,1])\xrightarrow{d_\sigma}...$$

where d_σ is de Rham differential taken along the fibers of σ, and $\Lambda^*_\sigma(M\times[0,1])$ denotes the fiberwise differential forms.

CLAIM: Any smooth family of closed forms $\alpha_t\in\Lambda^*(M)$, $t\in[0,1]$, represents an element $[\alpha_t]\in H^i(\mathbb{R}^\sigma_{M\times[0,1]})$. **There exists a smooth family $\eta_t\in\Lambda^*(M)$ such that $d\eta_t=\alpha_t$ if and only if $[\alpha_t]=0$.** ■

A smooth choice of anti-differential (3)

THEOREM: Let $\alpha_t \in \Lambda^k(M)$, $t \in [0,1]$ be a smooth family of exact forms. Then **there exists a smooth family of forms $\eta_t \in \Lambda^{k-1}(M)$, $t \in [0,1]$, such that $d\eta_t = \alpha_t$.**

Proof using the sheaf theory. Step 1 The proof will follow if we prove that **the family α_t represents 0 in the cohomology of $\mathbb{R}^\sigma_{M\times[0,1]}$.**

Step 2: Let $R^*\sigma_*$ denote the functor of derived direct image of the sheaf. This is the same as the functor mapping a sheaf to its fiberwise cohomology sheaf.

Let $\mathbb{R}_{M\times[0,1]}$ denote the constant sheaf on $M \times [0,1]$. Since $\mathbb{R}^\sigma_{M\times[0,1]} = \mathbb{R}_{M\times[0,1]} \otimes_{\mathbb{R}} \mathbb{R}^\sigma_{M\times[0,1]}$, and the functor $\otimes_{\mathbb{R}}\mathbb{R}^\sigma_{M\times[0,1]}$ is exact, one has

$$R^i\sigma_*(\mathbb{R}^\sigma_{M\times[0,1]}) = R^i\sigma_*(\mathbb{R}_{M\times[0,1]}) \otimes_{\mathbb{R}} \mathbb{R}^\sigma_{M\times[0,1]}$$

("the base change theorem"). Here $R^i\sigma_*(\mathbb{R}^\sigma_{M\times[0,1]})$ is the constant sheaf on $[0,1]$ with fiber $H^i(M,\mathbb{R})$. Therefore, a fiberwise exact form α_t represents zero in $R^i\sigma_*(\mathbb{R}^\sigma_{M\times[0,1]})$, and in

$$H^0(R^i\sigma_*(\mathbb{R}^\sigma_{M\times[0,1]})) = H^i(\mathbb{R}^\sigma_{M\times[0,1]}).$$

■