Symplectic geometry

lecture 3: Flow of diffeomorphisms and Moser isotopy lemma

Misha Verbitsky

HSE, room 306, 16:20,

September 11, 2021

Flow of diffeomorphisms

DEFINITION: Let $f: M \times [a,b] \longrightarrow M$ be a smooth map such that for all $t \in [a,b]$ the restriction $f_t := f\Big|_{M\times\{t\}} : M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c} : C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c}f$. **Then $f \longrightarrow (V_t^{-1})^*\frac{d}{dt}V_t^* f$ is a derivation** (that is, a vector field).

Proof: $\frac{d}{dt}V_t^*(fg) = V_t^*(f)\frac{d}{dt}V_t^* g + \frac{d}{dt}V_t^* f V_t^*(g)$ by the Leignitz rule, giving

$$(V_t^{-1})^*\frac{d}{dt}V_t^*(fg) = f\cdot(V_t^{-1})^*\frac{d}{dt}V_t^* g + g\cdot(V_t^{-1})^*\frac{d}{dt}V_t^* f.$$

■

DEFINITION: The vector field $V_t^{-1}\frac{d}{dt}V_t|_{t=c}$ is called **the vector field tangent to a flow of diffeomorphisms V_t at $t=c$.**

Flow of diffeomorphisms obtained from vector fields

EXERCISE: Let M be a compact manifold, and $\Psi: C^\infty M \longrightarrow C^\infty M$ is a ring automorphism. Prove that **Ψ is induced by an action of a diffeomorphism of M.**

THEOREM: Let M be a compact manifold, and $X_t \in TM$ a family of vector fields smoothly depending on $t \in [0, a]$. **Then there exists a unique diffeomorphism flow V_t, $t \in [0, a]$, such that $V_0 = \mathrm{Id}$ and $(V_t^{-1})^* \frac{d}{dt} V_t^* = X_t$.**

Proof. Step 1: Given $f \in C^\infty M$, we can solve an equation $\frac{d}{dt} W_t(f) = \operatorname{Lie}_{X_t}(f)$ (here $\operatorname{Lie}_{X_t}(f)$ denotes the derivative along the vector field). The solution $W_t(f)$ exists for all $t \in [0, x]$ and is unique by Peano theorem on existence and uniqueness of solutions of ODE.

Step 2: Since

$$\frac{d}{dt} W_t(fg) = \operatorname{Lie}_{X_t}(f) g + \operatorname{Lie}_{X_t}(g) f = \frac{d}{dt}(W_t(f) W_t(g)),$$

W_t is multiplicative. Also, it is invertible. Applying the previous exercise, we obtain that W_t is a diffeomorphism. ■

For another proof see Chapter 5 of *Arnold V.I., Ordinary Differential Equations*

Diffeomorphism flow and vector fields

DEFINITION: Let v_t be a vector field on M, smoothly depending on the "time parameter" $t \in [a,b]$, and $\Psi_t^{-1}\Psi_t : M \times [a,b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $\frac{d}{dt}\Psi_t = v_t$ for each $t \in [a,b]$, and $V_0 = \operatorname{Id}$. Then Ψ_t is called **an exponent of v_t**.

CLAIM: Let Ψ_t be an exponent of a time-dependent vector field v_t. **Then for any differenial form η, we have**

$$\left(\Psi^{-1}\right)^* \frac{d\Psi_t^*}{dt}\Big|_{t=t_0}\eta = \operatorname{Lie}_{v_t}\eta.$$

Proof: Both sides of this equation are derivations of de Rham algebra which commute with d and are equal to Lie_{vt} on functions. ■

COROLLARY 1: Let Ψ_t be an exponent of a time-dependent vector field v_t, and $\eta_t := \Psi_t^*\eta$. **Then $\Psi_a^*\eta = \eta + \int_0^a \operatorname{Lie}_{v_t}\eta_t dt$.** Conversely, if this equation holds, we have $\eta_t := \Psi_t^*\eta$. ■

Moser isotopy lemma (variant)

THEOREM: (Moser's isotopy lemma)
Let M be a compact symplectic manifold, and ω_t, $t \in [0,1]$ a smooth deformation of a symplectic form. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a diffeomorphism flow $\Psi_t \in \operatorname{Diff}(M)$ mapping ω_t to ω_0, for all t.**

Proof. Step 1: By Corollary 1, we need to find a family of vector fields v_t such that $\omega_0 + \int_0^a \operatorname{Lie}_{v_t} \omega_t dt = \omega_a$. Then $\omega_0 := \Psi_t^* \omega$. **This would follow if $\frac{d\omega_t}{dt} = \operatorname{Lie}_{v_t} \omega_t$.** Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. Then $\frac{d\omega_t}{dt} = d\eta_t$, where $\eta_t \in \Lambda^1(M)$. **This form can be chosen smoothly in t (Lecture 2).**

Step 2: By Step 1, to prove the theorem, we need to solve the equation

$$\operatorname{Lie}_{v_t} \omega_t = -\frac{d\omega_t}{dt}. \quad (*)$$

By Cartan's formula, $\operatorname{Lie}_{v_t} \omega_t = d(i_{v_t}(\omega_t))$. **Then (*) is equivalent to $d(i_{v_t}\omega_t) = -d\eta_t$.**

Step 3: The map $x \longrightarrow i_x(\omega)$ induces an isomorphism $TM \longrightarrow T^*M$ whenever ω is a non-degenerate 2-form. Therefore, there exists a unique $v_t \in TM$ such that $i_{v_t}\omega_t = -\eta_t$. This solves (*) and finishes the proof. ■

Darboux' theorem (reminder)

THEOREM: A symplectic manifold is locally symplectomorphic to a symplectic ball (in a neighbourhood of each point).

Proof. Step 1: It is sufficient to check that for any symplectic form ω_1 on $\mathbb{R}^n$ there exists a neighbourhood $U \ni 0$ such that (U, ω_1) is symplectomorphic to a symplectic ball.

Step 2: Choose coordinates x_i, y_i on $\mathbb{R}^{2n}$ in such a way that $\omega_1\Big|_{T_0\mathbb{R}^{2n}} = \omega_0\Big|_{T_0\mathbb{R}^{2n}}$, where $\omega_0 = \sum_i dx_i \wedge dy_i$. The form $\omega_t := t\omega_1 + (1-t)\omega_0$ is non-degenerate in 0, because $\omega_1|_0 = \omega_0|_0$. **Choose a starlike neighborhood $U \ni 0$ such that ω_t is non-degenerate for all $t \in [0,1]$.**

Step 3: In U the forms ω_t are all non-degenerate and cohomologous. As in the proof of Moser's lemma, **choose η_t such that $\frac{d\omega_t}{dt} = d\eta_t$, and a vector field $v_t := -\omega_t^{-1}(\eta_t)$, vanishing in 0.**

Substracting from η_t a constant 1-form, we may assume that $\eta_t\Big|_{T_0 U} = 0$. Then the the coefficients of the form η_t grow as $o(r)$, where r is the distance from zero. Therefore, for U sufficiently small, **the vector field Ψ_t integrates in the whole U, and defines a diffeomorphism Ψ between (U, ω_0) and $(\Psi(U), \omega_1)$.** Finally, since $v_t = 0$ in 0, the set $\Psi(U)$ contains 0. ∎

Symplectic structure on the total space of cotangent bundle

From now on, **the total space of the cotangent bundle** to M is denoted as T^*M. The **diffeomorphism group** of M is denoted $\operatorname{Diff}(M)$.

THEOREM: Let M be a smooth manifold. **Then T^*M is equipped with a natural, $\operatorname{Diff}(M)$-invariant symplectic form ω.**

Proof: Let $\pi: TM \longrightarrow M$ denote the projection. Consider a point $(x, \xi) \in T^*M$, where $x \in M$ and $\xi \in T_x^*M$. Let $\theta \in \Lambda^1(TM)$ be a 1-form which takes a tangent vector $v \in T_{(x,\xi)}T^*M$ and maps it to $\langle D\pi(v), \xi\rangle$. Here, $D\pi: T(T^*M) \longrightarrow TM$ is the differential, and $\langle\cdot,\cdot\rangle$ the pairing between T_xM and T_x^*M.

If we introduce the coordinates $p_1, ..., p_n$ on M and $q_1, ..., q_n$ on the fibers of $T^*M \longrightarrow M$ dual to $p_1, ..., p_n$, the form θ at a point $p_1, ..., p_n, q_1, ...q_n$ takes a vector $\sum_i f_i \frac{d}{dp_i} + \sum_i g_i \frac{d}{dpq_i}$ to $\sum_i f_i q_i$. Therefore, $\theta = \sum_i q_i dp_i$. **In the same coordinates $d\theta = \sum_i dq_i \wedge dp_i$; this form is symplectic and by construction $\operatorname{Diff}(M)$-invariant.** ■