Symplectic geometry

lecture 4: Hamiltonian symplectomorphisms

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Symplectic structure on the total space of cotangent bundle

From now on, **the total space of the cotangent bundle** to M is denoted as T^*M. The **diffeomorphism group** of M is denoted Diff(M).

THEOREM: Let M be a smooth manifold. **Then T^*M is equipped with a natural, Diff(M)-invariant symplectic form ω.**

Proof: Let $\pi : TM \longrightarrow M$ denote the projection. Consider a point $(x, \xi) \in T^*M$, where $x \in M$ and $\xi \in T^*_x M$. Let $\theta \in \Lambda^1(TM)$ be a 1-form which takes a tangent vector $v \in T_{(x,\xi)}T^*M$ and maps it to $\langle D\pi(v), \xi\rangle$. Here, $D\pi : T(T^*M) \longrightarrow TM$ is the differential, and $\langle \cdot, \cdot \rangle$ the pairing between T_xM and T^*_xM.

If we introduce the coordinates $p_1, ..., p_n$ on M and $q_1, ..., q_n$ on the fibers of $T^*M \longrightarrow M$ dual to $p_1, ..., p_n$, the form θ at a point $p_1, ..., p_n, q_1, ...q_n$ takes a vector $\sum_i f_i \frac{d}{dp_i} + \sum_i g_i \frac{d}{dpq_i}$ to $\sum_i f_i q_i$. Therefore, $\theta = \sum_i q_i dp_i$. **In the same coordinates $d\theta = \sum_i dq_i \wedge dp_i$; this form is symplectic and by construction Diff(M)-invariant.** ■

Lagrangian subspaces

REMARK: Let (V,ω) be a symplectic vector space, $\dim_{\mathbb{R}} V = 2n$, and $W \subset V$ be a subspace of dimension m. Denote by $W^\perp$ the symplectic orthogonal to W. Then $\frac{W}{W^\perp \cap W}$ is a symplectic vector space, hence $\operatorname{rk}(\omega|_W) = \dim W - \dim(W^\perp \cap W)$. Since $\dim W^\perp = 2n - \dim W$, the intersection $W^\perp \cap W$ has dimension at most n.

DEFINITION: A subspace $W \subset V$ is called
(i) **isotropic** if $\omega|_W = 0$; in this case $W \subset W^\perp$, hence $\dim W \leqslant n$.
(ii) **coisotropic** if $\omega|_{W^\perp} = 0$, or, equivalently, $W^\perp \subset W$. In this case $W \supset W^\perp$, hence $\dim W \geqslant n$.
(iii) **Lagrangian** if it is isotropic and coisotropic, that is, $W^\perp = W$.

REMARK: Isotropic and coisotropic subspaces **have minimal possible rank of ω for a given dimension.**

REMARK: An ω-orthogonal complement to an isotropic space is coisotropic, and vice versa.

Lagrangian submanifolds

DEFINITION: Let $X \subset M$ be a submanifold in a symplectic manifold. It is called **Lagrangian** if $T_x X \subset T_x M$ is Lagrangian for all $x \in X$.

THEOREM: Let $\xi \in \Lambda^1 M$ be a 1-form, and $\Gamma_\xi \subset T^* M$ its graph, considered as a submanifold in the total space of the cotangent bundle. **Then Γ_ξ is Lagrangian if and only if $d\xi = 0$.**

Proof: Let $\sigma : x \mapsto (x, \xi(x))$ be the standard diffeomorphism from M to Γ_ξ. Consider the restriction of θ to Γ_ξ. For each $u \in T_{(x,\xi(x))}\Gamma_\xi$, the form θ takes u to $\xi(d\pi(u))$. This implies that $\sigma^*\theta\Big|_{\Gamma_\xi} = \xi$, hence $\sigma^*\omega\Big|_{\Gamma_\xi} = d\xi$. ■

Hamiltonian vector fields

DEFINITION: Let $v \in TM$ be a vector field on a symplectic manifold (M, ω). We say that v is **symplectomorphic** if $\operatorname{Lie}_v \omega = 0$, that is, if ω is invariant under the corresponding diffeomorphism flow.

REMARK: From Cartan's formula, we have $\operatorname{Lie}_v \omega = d(i_v\omega)$, hence **$v$ is symplectomorphic if and only if the ω-dual 1-form is closed.**

DEFINITION: Let $v \in TM$ be a symplectomorphic vector field on a symplectic manifold (M, ω), and $\eta := i_v\omega$ the corresponding 1-form. We say that v is **a Hamiltonian vector field** if $i_v\omega$ is exact. Its **Hamiltonian** is a function f such that $df = i_v\omega$. The group of **Hamiltonian symplectomorphisms** is generated by diffeomorphisms obtained by exponents of a time-dependent vector field v_t, which is Hamiltonian for all $t \in [0,1]$.

REMARK: We have an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\ \delta\ } \operatorname{Ham}(M) \longrightarrow 0$$

If we identify $\operatorname{Ham}(M)$ with exact 1-forms, the differential $\delta: C^\infty(M) \longrightarrow \operatorname{Ham}(M)$ **is identified with the de Rham differential.**

The flux

DEFINITION: Let Ψ_t be a flow of symplectomorphisms of (M,ω), with $t\in[0,a]$ and $\Psi_0=\operatorname{Id}_M$. Given a homology class $[u]\in H_1(M,\mathbb{Z})$, consider its representative as a smooth circle $u:\ S^1\longrightarrow M$. Then $\Psi_t(u)$ takes $(s\in S^1, t\in [0,1])$ to $\Psi_t(u(s))$, giving a smooth map $\Psi_t(u):\ S^1\times[0,1]\longrightarrow M$. Denote its image by $\Psi_{[0,1]}(u)$. Define **the flux** of Ψ_t on u as $\int_{\Psi_{[0,1]}(u)}\omega$.

CLAIM: The flux of Ψ_t on u **is independent from the choice of u in $[u]$.**

Proof: Suppose that u_τ, $\tau\in[0,1]$ is a homotopy of u_0 to u_1. Consider the map $\tilde\Psi:\ S^1\times[0,1]\times[0,1]$ mapping (s,t,τ) to $\Psi_t(u_\tau)$. The surfaces $\Psi_{[0,1]}(u_0)$, $\Psi_{[0,1]}(u_1)$, $\cup_{\tau\in[0,1]}u_\tau$ and $\cup_{\tau\in[0,1]}\Psi_1(u_\tau)$ bound a solid torus $\operatorname{im}\tilde\Psi$, hence **to prove that $\int_{\Psi_{[0,1]}(u_1)}\omega=\int_{\Psi_{[0,1]}(u_2)}\omega$, it would suffice to show that $\int_{\cup_{\tau\in[0,1]}u_\tau}\omega=\int_{\cup_{\tau\in[0,1]}\Psi_a(u_\tau)}\omega$,** which is clear because Ψ_1 is a symplectomorphism.

To pass from homotopy to homology, we notice that flux is additive on unions of curves, hence **vanishes on a commutator of $\pi_1(M)$,** and defines a map from $\frac{\pi_1(M)}{[\pi_1(M),\pi_1(M)]}=H_1(M)$. ■

The flux conjecture

CLAIM: Flux of a Hamiltonian diffeomorphism vanishes.

Proof: Let Ψ_t be a Hamiltonian symplectomorphism flow on (M,ω), with $t \in [0,1]$ and $\Psi_t^{-1}\frac{d\Psi_t}{dt} = v_t$, where v_t is a Hamiltonian vector field with Hamiltonian H_t, and $u: \ S^1 \longrightarrow M$ a circle. To prove that the flux vanishes it would suffice to show that $\frac{d}{dt}\int_{\Psi_{[0,t]}(u)}\omega = 0$. This derivative can be computed as $\int_{\Psi_t(u))}\omega(\frac{d\Psi_t}{dt},\cdot)$. However, the 1-form $\omega(\frac{d\Psi_t}{dt},\cdot) = \Psi_t^*(dH_t)$ is exact, because v_t is Hamiltonian, hence $\frac{d}{dt}\int_{\Psi_{[0,t]}(u)}\omega = 0$. ■

REMARK: The famous "flux conjecture" states that vanishing of the flux is a necessary and sufficient condition: **if Ψ_t, $t \in [0,1]$ is a flow of symplectomorphisms, and the flux of Ψ_1 vanishes, then Ψ_1 is a Hamiltonian symplectomorphism.** Flux conjecture was proven by Kaoru Ono using Floer theory.

REMARK: Flux conjecture implies that **Hamiltonian symplectomorphisms is closed in the group of symplectomorphisms.**

Commutator of Hamiltonian vector fields

EXAMPLE: Let u_ε be a rotation of a 2-dimensional torus $S^1 \times S^1$ with the product metric which comes from unit circles and the symplectic structure given by the volume form. Consider a rotation along the first circle with angle ε. Then its flux through the second circle is the area of the segment bounded by the first circle and its image, that, is $2\pi\varepsilon$. In particular, **rotation of a torus is not a Hamiltonian symplectomorphism**.

THEOREM: Let X, Y be a Hamiltonian vector fields on (M, ω). **Then the commutator $[X, Y]$ is Hamiltonian.**

Proof: Let $f, g \in C^\infty M$, and let X_f, X_g be the corresponding Hamiltonian vector fields. Define **the Poisson bracket** as $\{f, g\} := \omega(X_f, X_g)$. By definition, $\omega(X_f, X_g) = \langle dg, X_f\rangle = \operatorname{Lie}_{X_f}(g)$, which gives $[X_f, X_g] = \operatorname{Lie}_{X_f}(X_g) = X_{\operatorname{Lie}_{X_f}(g)} = X_h$, where $h = \omega(X_f, X_g)$. Therefore, **commutator of Hamiltonian vector fields corresponds to the Poisson bracket on their Hamiltonian functions.** ■

Hamiltonian vector fields on Lagrangian fibrations

DEFINITION: Let (M,ω) be a symplectic manifold, and $\pi: M \longrightarrow X$ a smooth submersion. It is called **a Lagrangian fibration** if all its fibers are Lagrangian.

CLAIM: Let $\pi: M \longrightarrow X$ be a Lagrangian fibration, and H a function on X. Then **the corresponding Hamiltonian vector field v is tangent to the fibers of π. ** Moreover, **v is non-degenerate everywhere on a fiber $\pi^{-1}(x)$ if and only if $dH \neq 0$ in x;** otherwise $v|_{\pi^{-1}(x)} = 0$.

Proof: Let $L := \pi^{-1}(x)$. Consider ω as a map from T_mM to T^*_mM. Then ω^{-1} takes 1-forms vanishing in T_mL to vectors $v \in T_mM$ such that $\omega(v, \cdot)$ vanishes on T_mL. However, $T_mL^\perp = T_mL$ because L is Lagrangian. **Therefore, ω^{-1} takes 1-forms vanishing on TL to the vector fields tangent to L.**

The forms vanishing on TL are generated by $\pi^*\Lambda^1 X$, hence the corresponding Hamiltonian vector fields are tangent to the fibers of π. ■

REMARK: Let $L = \pi^{-1}(x)$ be a fiber of a Lagrangian fibration. Then ω defines a non-degenerate pairing between T_mL and T_xX. This implies that **the bundle TL is trivial.**

Collections of commuting Hamiltonians

REMARK: Let $\pi : M \longrightarrow X$ be a Lagrangian fibration, and v_1, v_2 two Hamiltonian vector fields obtained from $H_1, H_2 \in C^\infty X$ as above. **Then H_1, H_2 commute.** Indeed, the Poisson commutator $\{H_1, H_2\}$ is expressed as $\omega(v_t, u_t)$, vanishing for the vector fields tangent to fibers of π.

PROPOSITION: Let $\pi : M \longrightarrow X$ be a Lagrangian fibration, and L its fiber. Then **the bundle TL has a basis of globally defined commuting vector fields.** Moreover, **if L is compact, it is a torus.**

Proof: Let $L = \pi^{-1}(x)$ be a fiber, and $H_1, ..., H_n \in C^\infty X$ be a collection of functions such that $dH_1, ..., dH_n|_{T_x} X$ is a basis in $T^*_x X$. Then **the corresponding vector fields commute and define a basis in TL**. If these vector fields are **complete** (that is, can be integrated to a flow for all $t \in \mathbb{R}$), the manifold L is equipped with a transitive $\mathbb{R}^n$-action, which gives $L = \mathbb{R}^n/\Lambda$, where Λ is a discrete subgroup. If L is compact, Λ is a cocompact lattice, hence L is a torus. ■