Symplectic geometry

lecture 5: Arnold Conjecture

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Lagrangian submanifolds (reminder)

DEFINITION: Let $X \subset M$ be a submanifold in a symplectic manifold (M, ω). It is called **Lagrangian** if $T_x X \subset T_x M$ is Lagrangian for all $x \in X$. That is to say, $\dim X = 1/2 \dim M$ and $\omega|_X = 0$.

THEOREM: Let $\xi \in \Lambda^1 M$ be a 1-form, and $\Gamma_\xi \subset T^* M$ its graph, considered as a submanifold in the total space of the cotangent bundle. **Then Γ_ξ is Lagrangian if and only if $d\xi = 0$.**

Proof: Let $\sigma : x \mapsto (x, \xi(x))$ be the standard diffeomorphism from M to Γ_ξ. Consider the restriction of θ to Γ_ξ. For each $u \in T_{(x,\xi(x))}\Gamma_\xi$, the form θ takes u to $\xi(d\pi(u))$. This implies that $\sigma^*\theta\Big|_{\Gamma_\xi} = \xi$, hence $\sigma^*\omega\Big|_{\Gamma_\xi} = d\xi$. ■

Hamiltonian vector fields (reminder)

DEFINITION: Let $v \in TM$ be a vector field on a symplectic manifold (M, ω). We say that v is **symplectomorphic** if $\operatorname{Lie}_v \omega = 0$, that is, if ω is invariant under the corresponding diffeomorphism flow.

REMARK: From Cartan's formula, we have $\operatorname{Lie}_v \omega = d(i_v\omega)$, hence **$v$ is symplectomorphic if and only if the ω-dual 1-form is closed.**

DEFINITION: Let $v \in TM$ be a symplectomorphic vector field on a symplectic manifold (M, ω), and $\eta := i_v\omega$ the corresponding 1-form. We say that v is **a Hamiltonian vector field** if $i_v\omega$ is exact. Its **Hamiltonian** is a function f such that $df = i_v\omega$. The group of **Hamiltonian symplectomorphisms** is generated by diffeomorphisms obtained by exponents of a time-dependent vector field v_t, which is Hamiltonian for all $t \in [0, 1]$.

REMARK: We have an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\ \delta\ } \operatorname{Ham}(M) \longrightarrow 0$$

If we identify $\operatorname{Ham}(M)$ with exact 1-forms, the differential $\delta: C^\infty(M) \longrightarrow \operatorname{Ham}(M)$ **is identified with the de Rham differential.**

Hamiltonian vector fields on Lagrangian fibrations (reminder)

DEFINITION: Let (M,ω) be a symplectic manifold, and $\pi: M \longrightarrow X$ a smooth submersion. It is called **a Lagrangian fibration** if all its fibers are Lagrangian.

CLAIM: Let $\pi: M \longrightarrow X$ be a Lagrangian fibration, and H a function on X. Then **the corresponding Hamiltonian vector field v is tangent to the fibers of π.** Moreover, **v is non-degenerate everywhere on a fiber $\pi^{-1}(x)$ if and only if $dH \neq 0$ in x;** otherwise $v|_{\pi^{-1}(x)} = 0$.

Proof: Let $L := \pi^{-1}(x)$. Consider ω as a map from $T_m M$ to $T^*_m M$. Then ω^{-1} takes 1-forms vanishing in $T_m L$ to vectors $v \in T_m M$ such that $\omega(v,\cdot)$ vanishes on $T_m L$. However, $T_m L^\perp = T_m L$ because L is Lagrangian. **Therefore, ω^{-1} takes 1-forms vanishing on TL to the vector fields tangent to L.**

The forms vanishing on TL are generated by $\pi^*\Lambda^1 X$, hence the corresponding Hamiltonian vector fields are tangent to the fibers of π. ■

REMARK: Let $L = \pi^{-1}(x)$ be a fiber of a Lagrangian fibration. Then ω defines a non-degenerate pairing between $T_m L$ and $T_x X$. This implies that **the bundle TL is trivial.**

Hamiltonian isotopy of Lagrangian submanifolds

Let L_0, L_1 be Lagrangian submanifolds in (M,ω). We say that L_0 and L_1 are **Hamiltonian isotopic** if there exists a flow Ψ_t of Hamiltonian symplectomorphisms, $\Psi_0 = \mathrm{Id}$, $t \in [0,1]$, such that Ψ_1 maps L_1 to L_0.

EXAMPLE: Let ω be the standard symplectic form on the total space T^*M, and $\pi : T^*M \longrightarrow M$ the corresponding Lagrangian projection. Consider an exact 1-form η on M, and let $L_\eta \subset T^*M$ be its graph, considered as a Lagrangian submanifold. Let $H \in C^\infty M$, with $\eta = dH$. **The Hamiltonian vector field v associated with H is tangent to fibers of π and acts as a translation along each fiber.** Evaluating v at a fiber $\pi^{-1}(x) = T^*_x M$, we obtain that $v|_x = \eta|_x$, because $\omega(v,\cdot) = \eta$. Then $e^v = \eta$, hence **the Hamiltonian flow associated with H takes the zero section of T^*M to L_η.**

Darboux' theorem

THEOREM: A symplectic manifold is locally symplectomorphic to a symplectic ball (in a neighbourhood of each point).

Proof. Step 1: It is sufficient to check that for any symplectic form ω_1 on $\mathbb{R}^n$ there exists a neighbourhood $U \ni 0$ such that (U, ω_1) is symplectomorphic to a symplectic ball.

Step 2: Choose coordinates x_i, y_i on $\mathbb{R}^{2n}$ in such a way that $\omega_1\Big|_{T_0\mathbb{R}^{2n}} = \omega_0\Big|_{T_0\mathbb{R}^{2n}}$, where $\omega_0 = \sum_i dx_i \wedge dy_i$. The form $\omega_t := t\omega_1 + (1-t)\omega_0$ is non-degenerate in 0, because $\omega_1|_0 = \omega_0|_0$. **Choose a starlike neighborhood $U \ni 0$ such that ω_t is non-degenerate for all $t \in [0,1]$.**

Step 3: In U the forms ω_t are all non-degenerate and cohomologous. As in the proof of Moser's lemma, **choose η_t such that $\frac{d\omega_t}{dt} = d\eta_t$, and a vector field $v_t := -\omega_t^{-1}(\eta_t)$, vanishing in 0.**

Substracting from η_t a constant 1-form, we may assume that $\eta_t\Big|_{T_0U} = 0$. Then the the coefficients of the form η_t grow as $o(r)$, where r is the distance from zero. Therefore, for U sufficiently small, **the vector field Ψ_t integrates in the whole U, and defines a diffeomorphism Ψ between (U, ω_0) and $(\Psi(U), \omega_1)$.** Finally, since $v_t = 0$ in 0, the set $\Psi(U)$ contains 0. ■

Weinstein neighbourhood theorem

The following result is proven in the same way as the Darboux' theorem.

THEOREM: Let $X \subset M$ be a compact Lagrangian submanifold in (M, ω). Then **there exists a neighbourhood U of $X \subset M$ which is symplectomorphic to a neighbourhood of X in $X \subset T^*X$.**

Proof. Step 1: Consider a smooth retraction (say, orthogonal projection) $\pi : \ U \longrightarrow X$. Since X is Lagrangian, ω induces a non-degenerate pairing between TX and the fiberwise tangent bundle $T_\pi U$. This gives a natural isomorphism $T_\pi U|_X \cong T^*X$. Using the fiberwise exponent map, we obtain a diffeomorphism Ψ between U and a neighbourhood of zero in $T_\pi U|_X = T^*X$. **This diffeomorphism is compatible with the symplectic structure on $TU|_X$.**

Step 2: Let ω_0 be the symplectic structure on U induced from the embedding $U \longrightarrow T^*X$, and ω_1 the symplectic structure induced from M. Consider the form $\omega_t := t\omega_1 + (1-t)\omega_0$, where $t \in [0,1]$. Since $\omega_0\Big|_{TU|_X} = \omega_1\Big|_{TU|_X}$, **in a sufficiently small neighbourhood of X all ω_t are non-degenerate.** Shrinking U if necessarily, we may assume that this is true on all U.

Weinstein neighbourhood theorem (2)

THEOREM: Let $X \subset M$ be a compact Lagrangian submanifold in (M, ω). Then **there exists a neighbourhood U of $X \subset M$ which is symplectomorphic to a neighbourhood of X in $X \subset T^*X$.**

Step 3: Since U is diffeomorphic to X, and X is Lagrangian, all ω_t are exact. Therefore we may choose a smooth family $\eta_t \in \Lambda^1 U$ of 1-forms such that $d\eta_t = \omega_t$. Denote by $j: X \longrightarrow U$ the tautological embedding. Since $j^*(\omega_t) = 0$, one has $d(j^*\eta_t) = 0$. Replacing η_t by π^*(closed 1-form) if necessarily, we may assume that $j^*\eta_t$ is exact. Let $f_t \in C^\infty X$ be a family of functions which satisfy $df_t = j^*\eta_t$. Replacing η_t by $\eta_t - d(\pi^* f_t)$, **we obtain a family of 1-forms η_t which satisfy $d\eta_t = \omega_t$ and $j^*\eta_t = 0$.**

Step 4: Let $v_t := \omega_t^{-1}(\eta_t)$, and let Ψ_t be the corresponding diffeomorphism flow. Using Moser isotopy argument, we obtain that $\Psi_t^* \omega_0 = \omega_t$. However, ω_t^{-1} maps the kernel of the restiction map $\Lambda^1(U)|_X \longrightarrow \Lambda^1 X$ to $TX \subset TU|_X$, because X is Lagrangian. Therefore, v_t preserves X_t. Putting a metric on U, we may decompose the tangent bundle as $TU = T_\pi U \oplus T_\pi U^\perp$. Let v_t^π be the part of v_t which lies in $T_\pi U$. Since v_t^π vanishes in X, we have $|v_t^\pi| = o(r)$, where r is the distance to X. Using this estimate, it is easy to see that v_t **integrates to a diffeomorphism flow in a sufficiently small neighbourhood of X and maps it to another neighbourhood of X.** ∎

The flux of isotopic Lagrangian submanifolds

DEFINITION: Let $X \subset M$ be a compact Lagrangian submanifold, and $U \supset X$ its Weinstein neighbourhood, identified with an open subset of T^*X. We say that a sequence $X_i \subset U$ of Lagrangian submanifolds **converges to X in C^∞-topology** if each X_i is given as a graph of a closed 1-form η_i, and η_i converge to zero with all derivatives.

DEFINITION: Define **the flux** on the space of Lagrangian submanifolds with C^∞-topology as follows. Consider two isotopic (bot not necessarily Hamiltonian isotopic) submanifolds L_0, L_1, a homology class $[u] \in H_1(M,\mathbb{Z})$, and homotopic circles $u_0 \subset L_0$ an $u_1 \subset L_1$ representing u, denote by $\psi : S^1 \times [0,1] \longrightarrow M$ the homotopy map. Then $\operatorname{Flux}_u(L_0, L_1) := \int_{S^1\times[0,1]} \psi^*\omega$. Let $\operatorname{Flux}(L_0, L_1)$ be the associated map from $H_1(M,\mathbb{Z})$ to $\mathbb{R}$.

Flux: correctness of the definition

CLAIM: Suppose that ω is exact. Then the map $\operatorname{Flux}_u(L_0, L_1)$ **is independent from the choice of representatives u_0, u_1 and the homotopy u_t.**

Proof: Consider a class of the cohomology of the pair $H^2(M, L_1 \cup L_2)$ represented by ω, and let $\delta: H^1(L_1 \cup L_2) \longrightarrow H^2(M, L_1 \cup L_2)$ be the coboundary map of the exact sequence

$$... \longrightarrow H^1(L_1 \cup L_2) \xrightarrow{\delta} H^2(M, L_1 \cup L_2) \longrightarrow H^2(M) \longrightarrow H^2(L_1 \cup L_2) \longrightarrow ...$$

Since ω is exact, $\omega = \delta(\rho)$, and flux is obtained as $\operatorname{Flux}_u(L_0, L_1) = \int_{u_0 \cup -u_1} \rho$.
■

THEOREM: Let L_0, L_1 be Lagrangian submanifolds in (M, ω). Suppose that L_1 is a identified with a graph of a closed 1-form η in a Weinstein neighbourhood $U \subset L_0$. **Then L_1 is Hamiltonian isotopic to L_0 if and only if $\operatorname{Flux}(L_0, L_1)$.** Moreover, **every element of $\operatorname{Hom}(H_1(M, \mathbb{Z}), \mathbb{R})$ is represented by a flux of a Lagrangian submanifold.**

The proof follows from Claim 1 on the next slide.

Hamiltonian isotopy of Lagrangian submanifolds in T^*X

Claim 1: Let η be a closed 1-form on M, and $L_\eta \subset T^*M$ be its graph, considered as a Lagrangian submanifold. **Then L_η is Hamiltonian isotopic to the zero section L_0 of T^*M if and only if η is exact.**

Proof: We have already obtained the Hamiltonian isotopy from exactness of η. To prove the converse implication, we need to show that L_η is not Hamiltonian isotopic to L_0 when η is not exact. **To prove this we compute $\operatorname{Flux}(L_0, L_\eta)$ and show that it is non-zero unless η is exact.**

Let Ψ_t map (x, ξ) to $(x, \xi + t\eta)$, and let v be the vector field tangent to this action, $v\big|_(x, \xi) = (0, \eta)$. This vector field acts on T^*M by symplectomorphisms because it is locally in M Hamiltonian.

To finish the proof, it suffices to show that $\operatorname{Flux}_{[u]}(L_0, L_\eta) = \int_{[u]} \eta$ for any $[u] \in H^1(M, \mathbb{Z})$.

Let u be its representative in the zero section of T^*M, with $u: S^1 \longrightarrow M \subset T^*M$. The corresponding annulus $A := \Psi_{[0,1]}(u)$ has boundaries in u and $\Psi_1(u)$. Using $d\theta = \omega$, we obtain $\int_A \omega = -\int_u \theta + \int_{\Psi_1(u)} \theta$. The first integral vanishes, because $\theta = 0$ on the zero section, and the second integral is equal to $\int_u \eta \neq 0$ because $\theta = \eta$ on L_η. We proved that L_η is not Hamiltonian isotopic to the zero section. ■

Arnold Conjecture

REMARK: Let η be an exact form, and $L_\eta \subset T^*M$ its graph, considered as a Lagrangian submanifold in T^*M. Denote by L_0 the zero section of T^*M. The Lagrangian submanifolds L_η and L_0 intersect transversally if and only if $\eta = dH$, where H is a Morse function. Clearly, **the number of intersection points $L_0 \cap L_\eta$ is equal to the number of critical points of a Morse function, hence $\#(L_0 \cap L_\eta) \geqslant \sum_i b_i(M)$.**

CONJECTURE: (Arnol'd, proven by Floer)
Let $L \subset T^*M$ be a Lagrangian subvariety which transversally intersects the zero section $L_0 \subset T^*M$. Suppose that L is Hamiltonian isotopic to L_0. **Then $\#(L_0 \cap L) \geqslant \sum_i b_i(M)$.**

Strong Arnold Conjecture

REMARK: Note that **the minimal number $m(M)$ of critical points of Morse functions on M is, generally speaking, strictly bigger than $\sum_i b_i(M)$.**

The strong Arnold conjecture is still open.

CONJECTURE: (strong Arnol'd conjecture)
Let $L \subset T^*M$ be a Lagrangian subvariety which transversally intersects the zero section $L_0 \subset T^*M$. Suppose that L is Hamiltonian isotopic to L_0. **Then** $\#(L_0 \cap L) \geqslant m(M)$.

CONJECTURE: (Arnol'd-Givental conjecture)
Let (M, ω) be a symplectic manifold, and L_1, L_2 Hamiltonian isotopic Lagrangian submanifolds, intersecting transversally. **Then** $\#(L_1 \cap L_2) \geqslant \sum_i \dim H^i(M, \mathbb{Z}/2)$.