

Symplectic geometry

lecture 6: Gromov capacity

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Volume and capacity (reminder)

DEFINITION: **Standard symplectic structure** on \mathbb{R}^{2n} with coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ is $\omega := \sum_i dp_i \wedge dq_i$ (“Darboux coordinates”).

DEFINITION: **A symplectic ball** of radius r is the standard ball

$$B^{2n} := \left\{ (p_1, \dots, p_n, q_1, \dots, q_n) \in \mathbb{R}^{2n} \mid \sum_i p_i^2 + q_i^2 < r^2 \right\}$$

equipped with the standard symplectic form $\omega := \sum_i dp_i \wedge dq_i$

DEFINITION: **Symplectic volume** of a symplectic manifold (M, ω) , $\dim_{\mathbb{R}} M = 2n$, is $\text{Vol}(M, \omega) := \int_M \omega^n$.

Symplectic capacity **is a way to distinguish diffeomorphic symplectic manifolds of the same volume.**

Gromov capacity (reminder)

DEFINITION: An (open) symplectic embedding is an open embedding of symplectic manifolds, symplectomorphic to its image.

DEFINITION: Let (M, ω) be a symplectic manifold, and r a supremum of radii of all symplectic balls of the same dimension, admitting a symplectic embedding to M . The number $\text{capa}(M, \omega) := \pi r^2$ is called **Gromov symplectic capacity** of M .

THEOREM: (Ekeland-Hofer)

Let φ be an oriented diffeomorphism of symplectic manifold. Then **φ is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.**

Proof: Later today. ■

Gromov Non-Squeezing Theorem (reminder)

DEFINITION: A **symplectic cylinder** C_r is $\mathbb{R}^{2n} \times B_r$, where \mathbb{R}^{2n} is equipped with the standard symplectic form $\sum_i dp_i \wedge dq_i$, and B_r is the standard symplectic ball of radius r in \mathbb{R}^2 .

THEOREM: (Gromov) **Symplectic capacity of a symplectic cylinder C_r is equal to πr^2 .**

Proof: Later in these lectures. ■

REMARK: The volume is not the only obstruction to symplectic embeddings. Indeed, **the volume of the symplectic cylinder is infinite.**

Related: This theorem is also called “Symplectic camel theorem”, or “Gromov Non-Squeezing Theorem”.



Ekeland-Hofer theorem

THEOREM: (Ekeland-Hofer)

Let φ be an oriented diffeomorphism of symplectic manifold. Then φ is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.

Today, the Ekeland-Hofer theorem will be deduced from its linear version.

THEOREM: Ekeland-Hofer, the linear version

Let $(V = \mathbb{R}^{2n}, \omega = \sum_i dp_i \wedge dq_i)$ be a symplectic vector space, and $\varphi : V \longrightarrow V$ an oriented linear map which preserves the Gromov capacity of all ellipsoids. Then φ is a symplectomorphism.

Proof: Later in this course. ■

Open-compact topology

Plan of the proof:

0. Define the topology on the maps and subsets.
1. Deduce Ekeland-Hofer from its linear version, by approximating a symplectomorphism in appropriate scale with its differential.
2. Prove the linear version.

DEFINITION: Let $\varphi_i : M \longrightarrow N$ be a sequence of maps of metric spaces, $\varphi : M \longrightarrow N$ a map. Suppose that for each compact $K \subset M$ one has $\lim_i \sup_{x \in K} d(\varphi(x), \varphi_i(x)) = 0$. In this situation we say that $\{\varphi_i\}$ **converges to φ in the compact-open topology**.

EXERCISE: Let $K \subset M$ be a compact subset, $U \subset N$ an open subset, and let $C(K, U)$ be the set of all maps $\varphi : M \longrightarrow N$ taking K to U . **Prove that $C(K, U)$ is the base of compact-open topology.**

REMARK: From this exercise it follows that **the compact-open topology is independent from the metric.**

Hausdorff metric

DEFINITION: Let $Z \subset M$ be a metric space. **An ε -neighbourhood** $Z(\varepsilon)$ of Z is a union of all ε -balls centered in Z . **Hausdorff metric** d_H on closed subsets M is defined as follows: $d_H(X, Y)$ is infimum of all ε such that $Y \subset X(\varepsilon)$ and $X \subset Y(\varepsilon)$.

Properties of Hausdorff metric:

1. Let M be a metrizable topological space, and \mathcal{C} the set of compact subsets of M . Then the topology induced by the Hausdorff metric on \mathcal{C} is independent from the choice of the metric on M . **(Prove it!)**
2. Let $\varphi_i : M \longrightarrow N$ be a sequence of continuous maps, and $\Gamma_{\varphi_i} \subset M \times N$ their graphs. Suppose that M, N are compact. Then the sequence $\{\varphi_i\}$ converges to $\varphi : M \longrightarrow N$ (in the compact-open topology) **if and only if Γ_{φ_i} converges to Γ_{φ} in the Hausdorff topology.**

REMARK: Further on, we will consider “the Hausdorff metric” on the set of open subsets of M with compact closure. **This is in fact a pseudometric:** two open subsets U, V with the same closure satisfy $d_H(U, V) = 0$.

Hausdorff distance and boundary

LEMMA: Let E_1, E_2 be bounded open convex subsets in \mathbb{R}^n and $U_i := \mathbb{R}^n \setminus E_i$. Then $d_H(E_1, E_2) \geq d_H(U_1, U_2)$.

Proof: Suppose that $d_H(U_1, U_2) \geq \varepsilon$. Let $x \in U_1$ such that $x \notin U_2(\varepsilon)$. This means that $d(x, U_2) \geq \varepsilon$, hence $B_\varepsilon(x) \supset E_2$. By Hahn-Banach separation theorem, there exists a hyperplane H passing through x such that all E_1 lies to one side of this hyperplane. Since the distance from $B_\varepsilon(x)$ to this hyperplane is ε , and $B_\varepsilon(x) \subset E_2$, this implies that $d_H(E_1, E_2) \geq \varepsilon$. ■

LEMMA: Let E_1, E_2 be bounded open convex subsets in \mathbb{R}^n and $\partial(E_1) := \overline{E_1} \setminus E_1$ denote **the boundary** of E_1 . **Then** $d_H(E_1, E_2) \geq d_H(\partial E_1, \partial E_2)$.

Proof: Whenever $d_H(E_1, E_2) < \varepsilon$, we also have $d_H(U_1, U_2) < \varepsilon$, where $U_i = \mathbb{R}^n \setminus E_i$, hence every point x on a boundary of E_1 satisfies $x \in E_2(\varepsilon) \cap U_2(\varepsilon) = \partial E_2(\varepsilon)$. ■

Hausdorff distance between convex sets and homothety

Claim 1: Let $E \subset \mathbb{R}^n$ be a convex set containing 0, and $\varepsilon > 0$. Define $\lambda U := \{x \in \mathbb{R}^{2n} \mid \lambda^{-1}x \in U\}$. **Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that any convex set E_1 satisfying $d_H(E, E_1) < \varepsilon$ also satisfies $(1 - \delta)E \subset E_1 \subset (1 + \delta)E$.**

Proof. Step 1: Define $u(\delta) : \mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$ as

$$u(\delta) := \inf_{x \in \partial E} \min[d(x, \partial(1 - \delta)E), d(x, \partial(1 + \delta)E)].$$

Since ∂E is compact, and $d(x, \partial(1 - \delta)E)$ is 1-Lipschitz as a function of x , the number $d(x, \partial(1 - \delta)E)$ reaches its minimum somewhere on ∂E , and $u(\delta)$ is positive. An infimum of 1-Lipschitz functions is 1-Lipschitz, hence u is 1-Lipschitz. **Therefore, there exists δ such that $u(\delta) < \varepsilon$.**

Step 2: For any convex sets $E, E_1 \subset \mathbb{R}^n$, the inequality $d_H(E, E_1) < \varepsilon$ implies $\partial E_1 \subset \partial E(\varepsilon)$, by the previous lemma. Step 1 implies that anything which lies in an ε -neighbourhood of ∂E belongs to the segment bounded by $\partial(1 - \delta)E$ and $\partial(1 + \delta)E$. This implies that $(1 - \delta)E \subset \partial E_1 \subset (1 + \delta)E$. Passing to convex hulls, we obtain $(1 - \delta)E \subset E_1 \subset (1 + \delta)E$. ■

Symplectic capacity and Hausdorff convergence

CLAIM: Let $U_i \subset \mathbb{R}^{2n}$ be a Cauchy sequence of bounded open subsets, containing 0, and U their limit in the Hausdorff metric. Assume that U is convex. **Then** $\lim_i \text{cap}_G(U_i) = \text{cap}_G(U)$, where $\text{cap}_G(U)$ denotes the Gromov capacity.

Proof: Let $\lambda U := \{x \in \mathbb{R}^{2n} \mid \lambda^{-1}x \in U\}$. By Claim 1, for any $\varepsilon > 0$, almost all elements of the sequence U_i contain $(1-\varepsilon)U$ and are contained in $(1+\varepsilon)U$. In this situation,

$$\sqrt{1-\varepsilon} \text{cap}_G(U) \leq \text{cap}_G(U_i) \leq \sqrt{1+\varepsilon} \text{cap}_G(U).$$

■

Now we can deduce Ekeland-Hofer theorem from its linear version.

THEOREM: Ekeland-Hofer, the linear version

Let $(V = \mathbb{R}^{2n}, \omega = \sum_i dp_i \wedge dq_i)$ be a symplectic vector space, and $\varphi : V \longrightarrow V$ an oriented linear map which preserves the Gromov capacity of all ellipsoids.

Then φ is a symplectomorphism.

Image of an ellipsoid

Recall that **C^n -topology** on functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is the topology of uniform convergence on compacts for the derivatives up to n -th.

Claim 2: Let $\varphi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sequence of diffeomorphisms converging to identity in C^2 -topology, and $E \subset \mathbb{R}^n$ an ellipsoid. **Then for i sufficiently big, all $\varphi_i(E)$ are convex.**

Proof: Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic function $u(x_1, \dots, x_n) = \sum x_i^2 a_i^2$ such that $E = u^{-1}([0, 1[)$, and $u_i := (\varphi_i^{-1})(u)$. Then $\varphi_i(E) = u_i^{-1}([0, 1[)$. Clearly, the C^2 -convergence of φ_i to identity implies the C^2 -convergence of u_i to u . The functions u_i are convex for i sufficiently big, because **the Hessian $\text{Hess}(u_i - u)$ uniformly converges to zero, and $\text{Hess}(u)$ is positive definite.** On the other hand, the preimage $u_i^{-1}([0, 1[)$ is convex if u_i is convex. ■

Ekeland-Hofer theorem deduced from its linear version

THEOREM: (Ekeland-Hofer)

Let φ be an oriented diffeomorphism of symplectic manifold. Then φ is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.

Proof: Locally, every symplectic manifold is symplectomorphic to a symplectic ball (Darboux). Therefore it would suffice to prove the following (weaker) form of this theorem.

THEOREM: Let $(B, \omega) \xrightarrow{\varphi} (\mathbb{R}^{2n}, \omega)$ be an open embedding, mapping a symplectic ball to \mathbb{R}^{2n} with the usual symplectic structure, mapping 0 to 0, and preserving the Gromov symplectic capacities of all convex subsets. **Then φ is a symplectomorphism.**

Proof. Step 1: Let $\lambda > 1$, and $\Gamma_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the homothety mapping v to λv . By conformal invariance of cap_G , the diffeomorphism $\varphi_\lambda : B \rightarrow \mathbb{R}^{2n}$, defined as $\varphi_\lambda(v) := \Gamma_\lambda(\varphi(\Gamma_\lambda^{-1}(v)))$ preserves the Gromov symplectic capacities. **If $\varphi(x) = \sum_{i=1}^{\infty} P_i(x)$ is the Taylor decomposition for φ , with P_i homogeneous polynomials of degree i , one has $\varphi_\lambda(x) = \sum_{i=1}^{\infty} \lambda^{i-1} P_i(x)$.**

Ekeland-Hofer theorem deduced from its linear version (2)

Step 1: Let $\varphi_\lambda(v) := \Gamma_\lambda(\varphi(\Gamma_\lambda^{-1}(v)))$. If $\varphi(x) = \sum_{i=1}^{\infty} P_i(x)$ is the Taylor decomposition for φ , we have $\varphi_\lambda(x) = \sum_{i=1}^{\infty} \lambda^{i-1} P_i(x)$.

Step 2: For any diffeomorphism $(B, \omega) \xrightarrow{\varphi} (\mathbb{R}^{2n}, \omega)$ and any ellipsoid $E \subset B$, **there exists $\lambda_0 > 0$ such that $\varphi_\lambda(E)$ is convex for any $\lambda > \lambda_0$** . Indeed, the second derivative of φ_λ tends to 0 as λ tends to infinity, hence for λ sufficiently big this map maps E to a convex set.

Step 3: In an open-compact topology, $\lim_{\lambda \rightarrow \infty} \varphi_\lambda$ is equal to the differential $\mathfrak{D} := D_0\varphi$. For each ellipsoid $E \subset B$, we have $\mathfrak{D}(E) = \lim_{\lambda \rightarrow \infty} \varphi_\lambda(E)$ (in the Hausdorff topology). For λ sufficiently big, the set $\varphi_\lambda(E)$ is convex by Claim 2, and on convex subsets, the function capa_G is continuous in the Hausdorff topology. **This gives**

$$\text{capa}_G(\mathfrak{D}(E)) = \lim_{\lambda \rightarrow \infty} \text{capa}_G(\varphi_\lambda(E)) = \text{capa}_G(E),$$

that is, \mathfrak{D} preserves the symplectic capacity.

Step 4: Using the linear Ekeland-Hofer, we obtain that $\mathfrak{D} = D_0\varphi$ is a symplectomorphism. Since the choice of 0 was arbitrary, the same argument proves that **the differential of φ preserves the symplectic form everywhere where it is defined.** ■