Symplectic geometry

lecture 7: Ekeland-Hofer theorem

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Gromov capacity (reminder)

DEFINITION: An (open) symplectic embedding is an open embedding of symplectic manifolds, symplectimorphic to its image.

DEFINITION: Let (M,ω) be a symplectic manifold, and r a supremum of radii of all symplectic balls of the same dimension, admitting a symplectic embedding to M. The number $\operatorname{capa}(M,\omega) := \pi r^2$ is called **Gromov symplectic capacity** of M.

THEOREM: (Ekeland-Hofer)
Let φ be an oriented diffeomorphism of symplectic manifold. Then **φ is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.**

Proof: Later today. ■

Ekeland-Hofer theorem (reminder)

THEOREM: (Ekeland-Hofer)
Let φ be an oriented diffeomorphism of symplectic manifold. Then φ **is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.**

Today, the Ekeland-Hofer theorem **will be deduced from its linear version.**

THEOREM: Ekeland-Hofer, the linear version
Let $(V = \mathbb{R}^{2n}, \omega = \sum_i dp_i \wedge dq_i)$ be a symplectic vector space, and $\varphi : V \longrightarrow V$ an oriented linear map which preserves the Gromov capacity of all ellipsoids. **Then φ is a symplectomorphism.**

Proof: Next lecture. ■

Hausdorff metric (reminder)

DEFINITION: Let $Z \subset M$ be a metric space. **An ε-neighbourhood** $Z(\varepsilon)$ of Z is a union of all ε-balls centered in Z. **Hausdorff metric** d_H on closed subsets M is defined as follows: $d_H(X,Y)$ is infimum of all ε such that $Y \subset X(\varepsilon)$ and $X \subset Y(\varepsilon)$.

Properties of Hausdorff metric:

1. Let M be a metrizable topological space, and $\mathcal{C}$ the set of compact subsets of M. Then the topology induced by the Hausdorff metric on $\mathcal{C}$ is independent from the choice of the metric on M. **(Prove it!)**

2. Let $\varphi_i : M \longrightarrow N$ be a sequence of continuous maps, and $\Gamma_{\varphi_i} \subset M \times N$ their graphs. Suppose that M, N are compact. Then the sequence $\{\varphi_i\}$ converges to $\varphi : M \longrightarrow N$ (in the compact-open topology) **if and only if Γ_{φ_i} converges to Γ_φ in the Hausdorff topology.**

REMARK: Further on, we will consider "the Hausdorff metric" on the set of open subsets of M with compact closure. **This is in fact a pseudometric:** two open subsets U, V with the same closure satisfy $d_H(U,V) = 0$.

Hausdorff distance and boundary

LEMMA: Let E_1, E_2 be bounded open convex subsets in $\mathbb{R}^n$ and $U_i := \mathbb{R}^n \backslash E_i$. Then $d_H(E_1, E_2) \geqslant d_H(U_1, U_2)$.

Proof: Suppose that $d_H(U_1, U_2) \geqslant \varepsilon$. Let $x \in U_1$ such that $x \notin U_2(\varepsilon)$. This means that $d(x, U_2) \geqslant \varepsilon$, hence $B_\varepsilon(x) \supset E_2$. By Hahn-Banach separation theorem, there exists a hyperplane H passing through x such that all E_1 lies to one side of this hyperplane. Since the distance from the farthest point of $B_\varepsilon(x)$ to this hyperplane is ε, and $B_\varepsilon(x) \subset E_2$, this implies that $d_H(E_1, E_2) \geqslant \varepsilon$. ■

LEMMA: Let E_1, E_2 be bounded open convex subsets in $\mathbb{R}^n$ and $\partial(E_1) := \overline{E}_i \backslash E_i$ denote **the boundary** of E_i. **Then $d_H(E_1, E_2) \geqslant d_H(\partial E_1, \partial E_2)$.**

Proof: Whenever $d_H(E_1, E_2) < \varepsilon$, we also have $d_H(U_1, U_2) < \varepsilon$, where $U_i = \mathbb{R}^n \backslash E_i$, hence every point x on a boundary of E_1 satisfies $x \in E_2(\varepsilon) \cap U_2(\varepsilon) = \partial E_2(\varepsilon)$. ■

Hausdorff distance between convex sets and homothety

Claim 1: Let $E \subset \mathbb{R}^n$ be a convex set containing 0, and $\varepsilon > 0$. Define $\lambda U := \{x \in \mathbb{R}^{2n} \ \mid\ \lambda^{-1}x \in U\}$. **Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that any convex E_1 satisfying $d_H(E, E_1) < \varepsilon$ also satisfies $(1-\delta)E \subset E_1 \subset (1+\delta)E$.**

Proof. Step 1: Define $u(\delta) : \mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$ as

$$u(\delta) := \inf_{x \in \partial E} \min[d(x, \partial(1-\delta)E), d(x, \partial(1+\delta)E)].$$

Since ∂E is compact, and $d(x, \partial(1-\delta)E)$ is 1-Lipschitz as a function of x, the number $d(x, \partial(1-\delta)E)$ reaches its minumum somewhere on ∂E, and $u(\delta)$ is positive. An infimum of 1-Lipschitz functions is 1-Lipschitz, hence u is 1-Lipschitz. **Therefore, there exists δ such that $u(\delta) < \varepsilon$.**

Step 2: The inequality $d_H(E, E_1) < \varepsilon$ implies $\partial E_1 \subset \partial E(\varepsilon)$, by the previous lemma. Step 1 implies that **anything which lies in an ε-neighbourhood of ∂E belongs to the segment bounded by $\partial(1-\delta)E$ and $\partial(1-\delta)E$).** This implies that $(1-\delta)E \subset \partial E_1 \subset (1+\delta)E$. Passing to convex hulls, we obtain $(1-\delta)E \subset E_1 \subset (1+\delta)E$. ■

Symplectic capacity and Hausdorff convergence (remider)

CLAIM: Let $U_i \subset \mathbb{R}^{2n}$ be a Cauchy sequence of bounded open subsets, containing 0, and U their limit in the Hausdorff metric. Assume that U is convex. **Then** $\lim_i \operatorname{capa}_G(U_i) = \operatorname{capa}_G(U)$**,** where $\operatorname{capa}_G(U)$ denotes the Gromov capacity.

Proof: Let $\lambda U := \{x \in \mathbb{R}^{2n} \ \ | \ \ \lambda^{-1} x \in U\}$. By Claim 1, for any $\varepsilon > 0$, almost all elements of the sequence U_i contain $(1-\varepsilon)U$ and are contained in $(1+\varepsilon)U$. In this situation,

$$\sqrt{1-\varepsilon}\operatorname{capa}_G(U) \leqslant \operatorname{capa}_G(U_i) \leqslant \sqrt{1+\varepsilon}\operatorname{capa}_G(U).$$

■

Now we can deduce Ekeland-Hofer theorem from its linear version.

THEOREM: Ekeland-Hofer, the linear version
Let $(V = \mathbb{R}^{2n}, \omega = \sum_i dp_i \wedge dq_i)$ be a symplectic vector space, and $\varphi : V \longrightarrow V$ an oriented linear map which preserves the Gromov capacity of all ellipsoids. **Then φ is a symplectomorphism.**

Image of an ellipsoid

Recall that **C^n-topology** on functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is the topology of uniform convergence on compacts for the derivatives up to n-th.

Claim 2: Let $\varphi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sequence of diffeomorphisms converging to identity in C^2-topology, and $E \subset \mathbb{R}^n$ an ellipsoid. **Then for i sufficiently big, all $\varphi_i(E)$ are convex.**

Proof: Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic function $u(x_1, ..., x_n) = \sum x_i^2 a_i^2$ such that $E = u^{-1}([0,1[)$, and $u_i := (\varphi_i^{-1})(u)$. Then $\varphi_i(E) = u_i^{-1}([0,1[)$. Clearly, the C^2-convergence of φ_i to identity implies the C^2-convergence of u_i to u. The functions u_i are convex for i sufficiently big, because **the Hessian $\operatorname{Hess}(u_i - u)$ uniformly converges to zero, and $\operatorname{Hess}(u)$ is positive definite.** On the other hand, the preimage $u_i^{-1}([0,1[)$ is convex if u_i is convex. ■

Ekeland-Hofer theorem deduced from its linear version (reminder)

THEOREM: (Ekeland-Hofer)
Let φ be an oriented diffeomorphism of symplectic manifold. Then **φ is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.**

Proof: Locally, every symplectic manifold is symplectomorphic to a symplectic ball (Darboux). Therefore it would suffice to prove the following (weaker) form of this theorem.

THEOREM: Let $(B,\omega) \xrightarrow{\varphi} (\mathbb{R}^{2n},\omega)$ be an open embedding, mapping a symplectic ball to $\mathbb{R}^{2n}$ with the usual symplectic structure, mapping 0 to 0, and preserving the Gromov symplectic capacities of all convex subsets. **Then φ is a symplectomorphism**.

Proof. Step 1: Let $\lambda > 1$, and $\Gamma_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the homothety mapping v to λv. By conformal invariance of capa_G, the diffeomorphism $\varphi_\lambda : B \longrightarrow \mathbb{R}^{2n}$, defined as $\varphi_\lambda(v) := \Gamma_\lambda(\varphi(\Gamma_\lambda^{-1}(v)))$ preserves the Gromov symplectic capacities. **If $\varphi(x) = \sum_{i=1}^\infty P_i(x)$ is the Taylor decomposition for φ, with P_i homogeneous polynomials of degree i, one has $\varphi_\lambda(x) = \sum_{i=1}^\infty \lambda^{i-1} P_i(x)$.**

Ekeland-Hofer theorem deduced from its linear version (2)

Step 1: **Let $\varphi_\lambda(v) := \Gamma_\lambda(\varphi(\Gamma_\lambda^{-1}(v)))$.** If $\varphi(x) = \sum_{i=1}^\infty P_i(x)$ is the Taylor decomposition for φ, we have $\varphi_\lambda(x) = \sum_{i=1}^\infty \lambda^{i-1} P_i(x)$.

Step 2: For any diffeomorphism $(B,\omega) \xrightarrow{\varphi} (\mathbb{R}^{2n},\omega)$ and any ellipsoid $E \subset B$, **there exists $\lambda_0 > 0$ such that $\varphi_\lambda(E)$ is convex for any $\lambda > \lambda_0$.** Indeed, the second derivative of φ_λ tends to 0 as λ tends to infinity, hence for λ sufficiently big this map maps E to a convex set.

Step 3: In an open-compact topology, $\lim_{\lambda\to\infty} \varphi_\lambda$ is equal to the differential $\mathfrak{D} := D_0\varphi$. For each ellipsoid $E \subset B$, we have $\mathfrak{D}(E) = \lim_{\lambda\to\infty} \varphi_\lambda(E)$ (in the Hausdorff topology). For λ sufficiently big, the set $\varphi_\lambda(E)$ is convex by Claim 2, and on convex subsets, the function capa_G is continuous in the Hausdorff topology. **This gives**

$$\operatorname{capa}_G(\mathfrak{D}(E)) = \lim_{\lambda\to\infty} \operatorname{capa}_G(\varphi_\lambda(E)) = \operatorname{capa}_G(E),$$

that is, $\mathfrak{D}$ preserves the symplectic capacity.

Step 4: Using the linear Ekeland-Hofer, we obtain that $\mathfrak{D} = D_0\varphi$ is a symplectomorphism. Since the choice of 0 was arbitrary, the same argument proves that **the differential of φ preserves the symplectic form everywhere where it is defined.** ■