Symplectic geometry

lecture 9: pseudo-holomorphic curves and calibrations

Misha Verbitsky

HSE, room 306, 16:20,

October 2, 2021

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Almost complex submanifolds

DEFINITION: Let (M,J) be an almost complex manifoldm and $Z\subset M$ a closed subvariety. We allow Z to be singular, but we want the non-singular part to have a finite Riemannian volume in a neighbourhood of any singular point (prove that the **finiteness of the volume is independent of the choice of a Riemannian metric)**. Suppose that $J(T_zZ)\subset T_zZ$ for every smooth point $z\in Z$. Then $Z\subset M$ is called **an almost complex subvariety**.

REMARK: A subvariety of a complex manifold **is almost complex if and only if it is complex,** that is, it can be given as the set of common solutions of a system of holomorphic equations. This result follows from Newlander-Nirenberg for smooth subvarieties, and needs a non-trivial argument for singular.

FACT: A general (non-integrable) almost complex manifold **does not admit almost complex subvarieties of dimension** $\dim_{\mathbb C}\geqslant 2$, even locally. For example, S^6 **with the standard non-integrable complex structure has no almost complex subvarieties** except 1-dimensional.

Almost complex symplectic manifolds

DEFINITION: Let (M, ω) be a symplectic manifold, and I an almost complex structure. We say that **I is compatible with the symplectic structure** if $g(x,y) := \omega(Ix,y)$ for some Riemannian form g.

REMARK: In the same way one defines **almost complex structures compatible with a non-degenerate 2-form**.

DEFINITION: **An almost Kähler manifold** is a manifold (M, ω, I) equipped with a symplectic form ω and an almost complex structure I compatible with ω.

THEOREM: Let (M, ω) be a manifold equipped with a non-degenerate skew-symmetric 2-form. Then **the space of almost complex structures compatible with ω is contractible.**

Space of almost complex structures compatible with ω

THEOREM 1: Let (M, ω) be a manifold equipped with a non-degenerate skew-symmetric 2-form. Then **the space C of almost complex structures compatible with ω is contractible.**

Proof. Step 1: We identify C with the space of Riemannian metrics g such that $g^{-1}\omega$ is an almost complex structure compatible with the ω. Since the space R of Riemannian metrics is convex, it is contractible, hence **to prove that C is conractible it would suffice to show that C is a deformational retract of R.**

Step 2: Let $A := g^{-1}\omega$, that is, $g(Ax, y) = \omega(x, y)$. **The matrix A us skew-symmetric:** $g(Ax, y) = \omega(x, y) = -\omega(y, x) = -g(Ay, x) = -g(x, Ay)$. A skew-symmetric matrix can be written in some orthonormal basis as

$$\omega = \begin{pmatrix} A = \begin{matrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & \alpha_n \\ -\alpha_n & 0 \end{matrix} \end{pmatrix}$$

Since $g(A^2x, x) = -g(Ax, Ax)$, **the matrix Ar is symmetric and negative definite. Then $B_t := e^{-\frac{t}{2}\log(-A^2)}$ is correctly defined, symmetric, and continuously depends on t and A.**

Space of almost complex structures compatible with ω (2)

Step 3: The operator AB_1 is written in the same coordinates as

$$\begin{pmatrix} \omega = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix},$$

that is, defines an almost complex structure on M. Since $\omega(AB_1x, y) = g(B_1x, y)$, and B_1 is symmetric, **this almost complex structure is compatible with ω.**

Step 4: The map $g, t \xrightarrow{\Psi_t} g(B_tx, y)$ for $t = 1$ gives the metric $g(B_1\cdot, \cdot) \in C$, and for $t = 0$ gives g. **Therefore, the map Ψ_t retracts C to R.** ■

Conformal structure

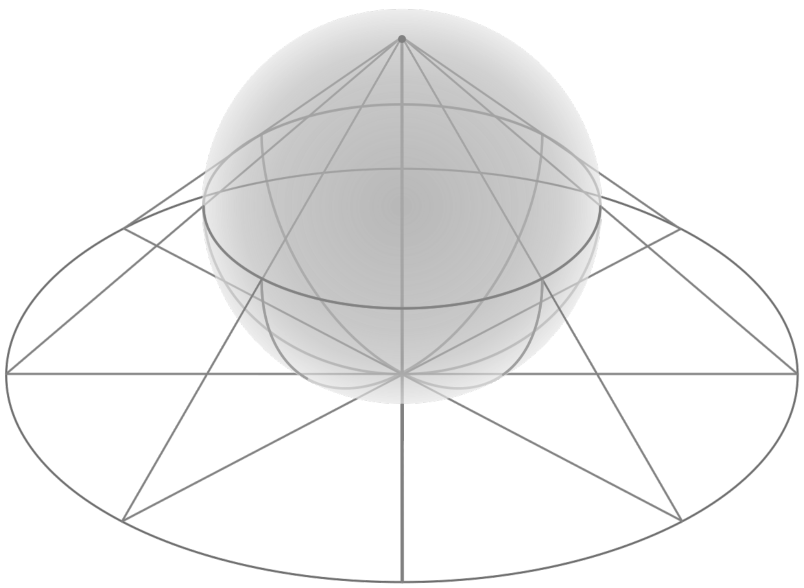
DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

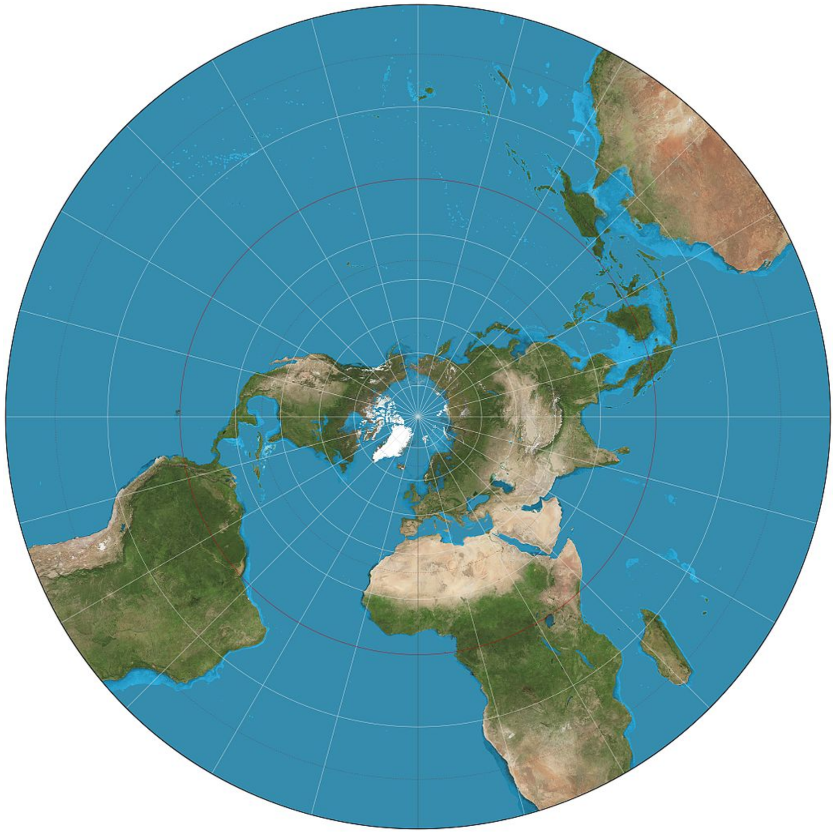
DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

EXERCISE: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Prove that h and h' are conformally equivalent**. Prove that any metric conformally equivalent to a Hermitian metric is Hermitian.

REMARK: The last statement is clear from the definition, and true in any dimension.

Stereographic projection

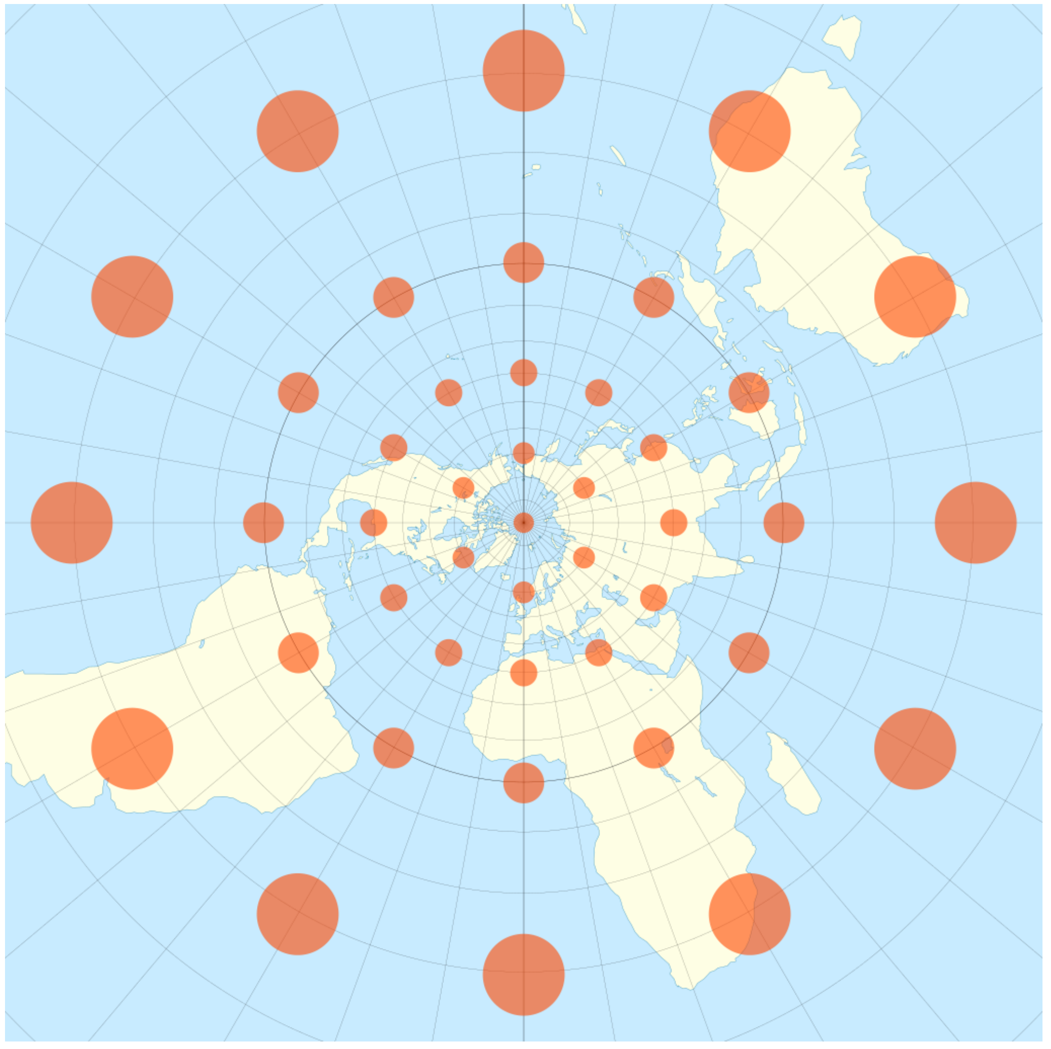




Stereographic projection is a light projection from the south pole to a plane tangent to the north pole.

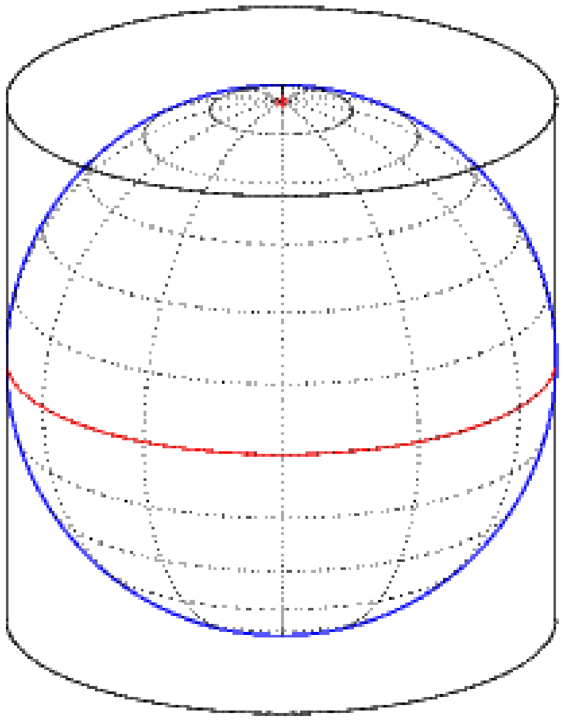
stereographic projection is conformal (prove it!)

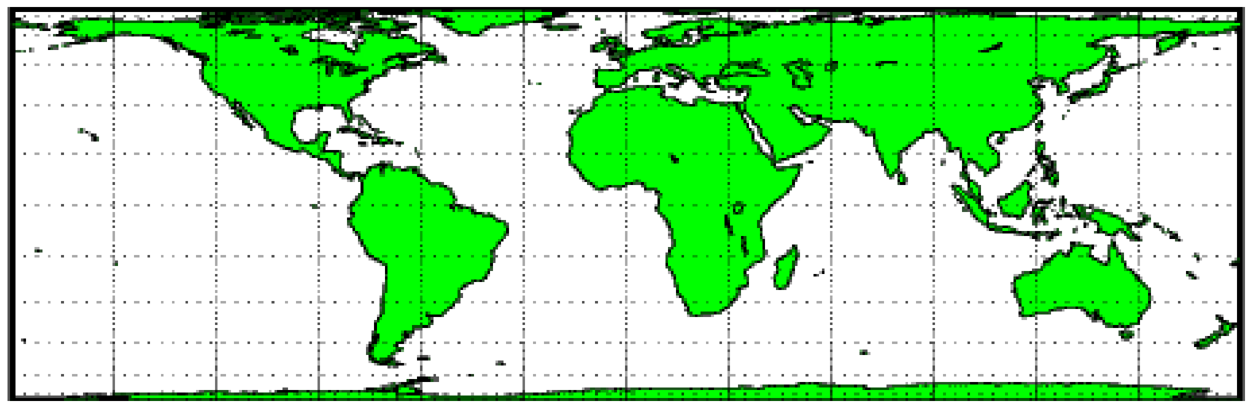
Stereographic projection (2)



The stereographic projection with Tissot's indicatrix of deformation.

Cylindrical projection





Cylindrical projection is not conformal. However, it is volume-preserving.

Conformal structures and almost complex structures

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1}\,)$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

EXERCISE: Prove that **a smooth map from one Riemann surface to another is holomorphic if and only if it preserves the conformal structure everywhere.**

Calibrations

DEFINITION: (Harvey-Lawson, 1982)
Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and $\operatorname{Vol}(W)$ denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any p-form $\eta \in \Lambda^p V$, let **comass** $\operatorname{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, ..., v_p)}{|v_1||v_2|...|v_p|}$, for all p-tuples $(v_1, ..., v_p)$ of vectors in V and **face** be the set of planes $W \subset V$ where $\frac{\eta}{\operatorname{Vol}(W)} = \operatorname{comass}(\eta)$.

DEFINITION: A **precalibration** on a Riemannian manifold is a differential form with comass $\leqslant 1$ everywhere.

DEFINITION: A **calibration** is a precalibration which is closed.

DEFINITION: Let η be a k-dimensional precalibration on a Riemannian manifold, and $Z \subset M$ a k-dimensional subvariety (we always assume that the Hausdorff dimension of the set of singular points of Z is $\leqslant k-2$, because in this case a compactly supported differential form can be integrated over Z). We say that Z is **calibrated by η** if at any smooth point $z \in Z$, the space $T_z Z$ is a face of the precalibration η.

H. Blaine Lawson, Jr.,
Berkeley, 1972

F. Reese Harvey,
Berkeley, 1968

Source: George M. Bergman, Berkeley

Calibrations and minimal submanifolds

REMARK: Clearly, for any precalibration η, one has

$$\operatorname{Vol}(Z) \geqslant \int_Z \eta, \qquad (*)$$

where $\operatorname{Vol}(Z)$ denotes the Riemannian volume of a compact Z, and the equality happens iff Z is calibrated by η. If, in addition, η is closed, the number $\int_Z \eta$ is a cohomological invariant. Then, **the inequality (*) implies that Z minimizes the Riemannian volume in its homology class.**

DEFINITION: A subvariety Z is called **minimal** if for any sufficiently small deformation Z' of Z in class C^1, one has $\operatorname{Vol}(Z') \geqslant \operatorname{Vol}(Z)$.

REMARK: Calibrated subvarieties are obviously minimal.

DEFINITION: An almost complex Hermitian manifold (M, I, h) with closed Hermitian form $\omega := h(I(\cdot), \cdot)$ is called **almost Kähler**.

EXAMPLE: (Wirtenger's inequality):
Let (M, I, ω) be an almost Kähler manifold. **Then $\frac{\omega^d}{d!2^d}$ is a calibration which calibrates d-dimensional almost complex subvarieties.** In patricular, **almost complex submanifolds in almost Kähler manifolds are minimal**.

Pseudoholomorphic curves

DEFINITION: Let (M, J) be an almost complex manifold, (Σ, I) a Riemann surface, and $\varphi: \Sigma \longrightarrow M$ an I-holomorphic map, that is, a smooth map with $D\varphi(Ix) = J(D\varphi(x))$. Then $\varphi(\Sigma)$ is called **a pseudo-holomorphic curve**, or **a J-holomorphic curve**.

THEOREM: (Wirtenger's inequality):
Let (M, I, ω) be an almost Kähler manifold. **Then $\frac{1}{2}\omega$ is a calibration which calibrates pseudo-holomorphic curves.**

Proof: Let g_S be the Riemannian volume form on S, and $x, y \in T_s S$ be orthogonal vectors of length 1. Then $g_S(x, y) = 1$ and $\omega(x, y) = g(x, Iy) \leqslant 1$, and the equality is realized if and only if $x = Iy$, by Cauchy-Bunyakovsky-Schwarz inequality. ■

COROLLARY: Pseudoholomorphic curves are minimal.

Symplectic capacity and the pseudoholomorphic curves

THEOREM 2: Let $M = \mathbb{C}P^1 \times T^{2n}$ be the product of $\mathbb{C}P^1$ and a torus, equipped with the standard symplectic structure, and J a compatible almost complex structure. **Then for any $x \in M$ there exists a pseudo-holomorphic curve S homologous to $\mathbb{C}P^1 \times \{m\}$ and passing through x.**

This theorem implies Gromov's non-squeezing theorem.

THEOREM: (Gromov) Symplectic capacity of a symplectic cylinder Cyl_1 is equal to π.