Symplectic geometry

lecture 10: Proof of Gromov's Non-Squeezing Theorem

Misha Verbitsky

HSE, room 306, 16:20,

October 6, 2021

Almost complex structures (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

DEFINITION: Let (M, ω) be a symplectic manifold, and I an almost complex structure. We say that **I is compatible with the symplectic structure** if $g(x,y) := \omega(Ix, y)$ for some Riemannian form g.

THEOREM 1: Let (M, ω) be a manifold equipped with a non-degenerate skew-symmetric 2-form. Then **the space C of almost complex structures compatible with ω is contractible.**

Calibrations (reminder)

DEFINITION: (Harvey-Lawson, 1982)
Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and Vol(W) denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any p-form $\eta \in \Lambda^p V$, let **comass** $\operatorname{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, ..., v_p)}{|v_1||v_2|...|v_p|}$, for all p-tuples $(v_1, ..., v_p)$ of vectors in V and **face** be the set of planes $W \subset V$ where $\frac{\eta}{\operatorname{Vol}(W)} = \operatorname{comass}(\eta)$.

DEFINITION: A **precalibration** on a Riemannian manifold is a differential form with comass $\leqslant 1$ everywhere.

DEFINITION: A **calibration** is a precalibration which is closed.

DEFINITION: Let η be a k-dimensional precalibration on a Riemannian manifold, and $Z \subset M$ a k-dimensional subvariety (we always assume that the Hausdorff dimension of the set of singular points of Z is $\leqslant k-2$, because in this case a compactly supported differential form can be integrated over Z). We say that Z is **calibrated by η** if at any smooth point $z \in Z$, the space $T_z Z$ is a face of the precalibration η.

Calibrations and minimal submanifolds (reminder)

REMARK: Clearly, for any precalibration η, one has

$$\operatorname{Vol}(Z) \geqslant \int_Z \eta, \qquad (*)$$

where $\operatorname{Vol}(Z)$ denotes the Riemannian volume of a compact Z, and the equality happens iff Z is calibrated by η. If, in addition, η is closed, the number $\int_Z \eta$ is a cohomological invariant. Then, **the inequality (*) implies that Z minimizes the Riemannian volume in its homology class.**

DEFINITION: A subvariety Z is called **minimal** if for any sufficiently small deformation Z' of Z in class C^1, one has $\operatorname{Vol}(Z') \geqslant \operatorname{Vol}(Z)$.

REMARK: **Calibrated subvarieties are obviously minimal.**

Pseudoholomorphic curves (reminder)

DEFINITION: Let (M, J) be an almost complex manifold, (Σ, I) a Riemann surface, and $\varphi : \Sigma \longrightarrow M$ an I-holomorphic map, that is, a smooth map with $D\varphi(Ix) = J(D\varphi(x))$. Then $\varphi(\Sigma)$ is called **a pseudo-holomorphic curve**, or **a J-holomorphic curve**.

THEOREM: (Wirtenger's inequality):
Let (M, I, ω) be an almost Kähler manifold. **Then $\frac{1}{2}\omega$ is a calibration which calibrates pseudo-holomorphic curves.**

Proof: Let g_S be the Riemannian volume form on S, and $x, y \in T_s S$ be orthogonal vectors of length 1. Then $g_S(x, y) = 1$ and $\omega(x, y) = g(x, Iy) \leqslant 1$, and the equality is realized if and only if $x = Iy$, by Cauchy-Bunyakovsky-Schwarz inequality. ■

COROLLARY: Pseudoholomorphic curves are minimal.

Symplectic capacity and the pseudoholomorphic curves

THEOREM 2: Let $M = \mathbb{C}P^1 \times T^{2n}$ be the product of $\mathbb{C}P^1$ and a torus, equipped with the standard symplectic structure, and J a compatible almost complex structure. **Then for any $x \in M$ there exists a pseudo-holomorphic curve S homologous to $\mathbb{C}P^1 \times \{m\}$ and passing through x.**

This theorem implies Gromov's non-squeezing theorem.

THEOREM: (Gromov) Symplectic capacity of a symplectic cylinder Cyl_1 **is equal to π.**

Proof of Gromov's theorem

THEOREM: (Gromov)
Symplectic capacity of a symplectic cylinder Cyl_1 is equal to π.

Proof. Step 1: Let $f_1: B_r \longrightarrow \mathrm{Cyl}_1$ be a symplectic embedding, $r > 1$, and I the usual (flat) almost complex structure on $B_r \subset \mathbb{C}^{n+1}$. Consider the manifold $M = \mathbb{C}P^1 \times T^{2n}$, equipped with the standard symplectic structure, and let $f_2: \mathrm{Cyl}_1 \longrightarrow \mathbb{C}P^1 \times T^{2n}$ be a symplectic map taking $\mathrm{Cyl}_1 = \Delta \times \mathbb{R}^{2n}$ to $\mathbb{C}P^1 \times T^{2n}$ applying the $\mathbb{Z}^{2n}$ quotient on the second argument and the natural symplectomorphism $\Delta \xrightarrow{\sim} \mathbb{C}P^1 \backslash \infty$ on the first argument.

Step 2: Choose the lattice $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n}$ in such a way that its fundamental domain contains $f_1(B_r)$. Then the composition $f_1 \circ f_2$ gives a symplectic embedding $B_r \longrightarrow M = \mathbb{C}P^1 \times T^{2n}$.

We obtained that **Gromov's non-squeezing theorem is deduced from the following result.**

Proof of Gromov's theorem (2)

THEOREM: Let $M = \mathbb{C}P^1 \times T^{2n}$ be equipped with the standard symplectic form, with the symplectic volume of $\mathbb{C}P^1$ equal to π, and $\varphi: \ B_r \longrightarrow M$ a symplectic embedding. **Then $r \leqslant 1$.**

Proof. Step 1: Choose a flat complex structure and the flat Hermitian metric on B_r. Denote by g_0 the corresponding Hermitian metric on $\varphi(B_r)$. Then g_0 can be extended to a Riemannian metric g_1 on M such that $g_0 = g_1$ in a ball $\varphi(B_{r-\varepsilon})$, for some ε such that $r - \varepsilon > 1$. The operation $g_1(\cdot, \cdot) \longrightarrow g_1(B_1 \cdot, \cdot)$ constructed in the proof of Theorem 1, gives a metric g compatible with the symplectic structure on M and coinciding with g_0 in a ball $\varphi(B_{r-\varepsilon})$. **Replacing B_r by $B_{r-\varepsilon}$, we can add to the assumptions of the theorem the following assumption.**

There exists a compatible almost complex structure such that uts restriction to $\varphi(B_r)$ is equal to the standard complex structure on $B_r \subset \mathbb{C}^{n+1}$.

Proof of Gromov's theorem (3)

THEOREM: Let $M = \mathbb{C}P^1 \times T^{2n}$ equipped with the standard symplectic form, with the symplectic volume of $\mathbb{C}P^1$ equal to π, and $\varphi: \ B_r \longrightarrow M$ a symplectic embedding. **Assume that there exists a compatible almost complex structure such that its restriction to $\varphi(B_r)$ is equal to the standard complex structure on $B_r \subset \mathbb{C}^{n+1}$. Then $r \leqslant 1$.**

Step 2: Let $x \in M$ be the image of the center of B_r, and $S \subset M$ the pseudo-holomorphic curve which passes through x by Theorem 2. Then $\pi = \int_S \omega_M \geqslant \int_{\varphi^{-1}(S)} \omega$, where ω_M is the symplectic form on M, and ω the symplectic form on B_r. Since S is pseudo-holomorphic, $\int_{\varphi^{-1}(S)} \omega_{B_r}$ **is the Riemannian volume of its intersection with $\varphi(B_r)$.**

Step 3: We obtained a complex curve $D := \varphi^{-1}(S)$ passing through 0 in a ball B_r with flat Riemannian metric and the standard complex structure, with the Riemannian volume $\operatorname{Vol}(D) \leqslant \pi$. Applying the homothety, we obtain a properly embedded complex disk in the ball B of radius 1, passing through 0 and with area $r^{-1}\pi$. **For any $r > 1$, this is impossible, as follows from the following statement,** proven later today.

PROPOSITION: Let $D \subset B_1$ be a closed complex disk in a unit ball $B_1 \subset \mathbb{C}^n$, with $0 \in D$. **Then $\operatorname{Vol}(D) \geqslant \pi$.**

Monotonicity formula (1)

PROPOSITION: Let $D \subset B_1$ be a complex curve (that is, a closed 1-dimensional complex subvariety) in a unit ball $B_1 \subset \mathbb{C}^n$, with $0 \in D$. **Then** $\operatorname{Vol}(D) \geqslant \pi$**.**

We deduce it from the **monotonicity formula.**

LEMMA: (monotonicity formula)
Let (B, η) be a unit ball $B_1(0) \subset \mathbb{R}^n$ equipped with a calibration $\eta \in \Lambda^k B_1$ with constant coefficients, and the standard flat metric, and $\Delta \subset B$ a calibrated k-dimensional submanifold passing through 0. We assume that Δ is immersed to B with its boundary in such a way that $\partial\Delta$ is mapped to ∂B. **Let Δ_t be the intersection of Δ with a ball $B_t(0)$ of radius t. Then the function $t \longrightarrow t^{-k}\operatorname{Vol}(\Delta_t)$ is non-decreasing.**

Clearly, $\lim\limits_{t\to 0} \operatorname{Vol}(\Delta_t) = \pi t^2$ (on a very small scale, any smooth disk is flat). **Then the monotonicity lemma gives $\operatorname{Vol}\Delta \geqslant \pi$.**

Monotonicity formula (2)

LEMMA: (monotonicity formula)
Let (B, η) be a unit ball equipped equipped with a calibration $\eta \in \Lambda^k B_1$ with constant coefficients, and $\Delta \subset B$ a calibrated k-dimensional submanifold passing through 0. **Let Δ_t be the intersection of Δ with a ball $B_t(0)$ of radius t. Then the function $t \longrightarrow t^{-k} \operatorname{Vol}(\Delta_t)$ is non-decreasing.**

Proof. Step 1: Let $C(\partial\Delta) \subset B$ be the cone obtained as a union of all intervals connecting 0 and points of $\partial\Delta \subset \partial B$. Then

$$\operatorname{Vol} C(\partial\Delta) \geqslant \int_{C(\partial\Delta)} \eta = \int_{\Delta} \eta = \operatorname{Vol} \Delta.$$

The first equality follows from the Stokes' theorem, because $\partial\Delta = \partial C(\partial\Delta)$, and the first inequality holds because η is a calibration.

Step 2: The formula for the cone volume gives $\operatorname{Vol} C(\partial\Delta) = \frac{1}{k} l(\partial\Delta)$, where $l(\partial\Delta)$ is the area of the boundary $\partial\Delta$. **This implies $\operatorname{Vol} \Delta \leqslant \frac{1}{k} l(\partial\Delta)$.**

Monotonicity formula (3)

LEMMA: (monotonicity formula)
Let (B,η) be a unit ball equipped equipped with a calibration $\eta\in\Lambda^k B_1$ with constant coefficients, and $\Delta\subset B$ a calibrated k-dimensional submanifold passing through 0. **Let Δ_t be the intersection of Δ with a ball $B_t(0)$ of radius t. Then the function $t\longrightarrow t^{-k}\operatorname{Vol}(\Delta_t)$ is non-decreasing.**

Step 2: We have obtained $\operatorname{Vol}\Delta\leqslant\frac{1}{k}l(\partial\Delta)$.

Step 3: The same argument applied to Δ_t gives $t^{-k}\operatorname{Vol}\Delta_t\leqslant\frac{1}{kt^{k-1}}l(\partial\Delta_t)$. (homothety maps the k-dimensional Riemann volume μ to $t^k\mu$). On the other hand, $\frac{d}{dt}\operatorname{Vol}\Delta_t\geqslant l(\partial\Delta_t)$, because the volume of the strip of Δ between $\partial\Delta_t$ and $\partial\Delta_{t+\varepsilon}$ is bounded by $\varepsilon l(\partial\Delta_{t+\varepsilon})$. This gives

$$\operatorname{Vol}\Delta_t\leqslant\frac{t}{k}l(\partial\Delta_t)\leqslant\frac{t}{k}\frac{d}{dt}\operatorname{Vol}\Delta_t.$$

Step 4: Let $f(t):=\operatorname{Vol}\Delta_t$. The last formula of Step 3 gives $f(t)\leqslant\frac{t}{k}f'(t)$, hence

$$\frac{d}{dt}t^{-k}f(t)=t^{-k}f'(t)-kt^{-k-1}f(t)\geqslant t^{-k}f'(t)-t^{-k}f'(t)=0.$$

■