Symplectic geometry

lecture 11: Proof of Gromov's Non-Squeezing Theorem (again)

Misha Verbitsky

HSE, room 306, 16:20,

October 9, 2021

Almost complex structures

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

DEFINITION: Let (M, ω) be a symplectic manifold, and I an almost complex structure. We say that **I is compatible with the symplectic structure** if $g(x, y) := \omega(Ix, y)$ for some Riemannian form g. In that case, g is called **compatible with I** as well.

LEMMA 1: Let (M, ω) be a symplectic manifold, and $(B_r, \sum dx_i \wedge dy_i) \hookrightarrow (M, \omega)$ a symplectic embedding. Then for any $\varepsilon > 0$ **there exists an almost complex structure I on (M, ω) compatible with ω and equal to the standard complex structure on $B_{r-\varepsilon} \subset B_r$.**

Proof: Let g_1 be any Riemannian metric on M, compatible with ω and g_0 the standard Riemannian metric on B_r. Denote by $\lambda \in C^\infty M$ the cut-off function vanishing outside of B_r and equal to 1 on $B_{r-\varepsilon}$. Then $\tilde{g} := \lambda g_0 + (1 - \lambda) g_1$ is equal to g_0 on $B_{r-\varepsilon}$ and g_1 outside of B_r. This form is compatible with ω in $B_{r-\varepsilon}$ and $M \backslash B_r$. To make it compatible with Ω everywhere, we use the argument from Theorem 1 in Lecture 9: produce a symmetric matrix $B_1 = e^{-\frac{1}{2}\log(-A^2)}$, where $A := g^{-1}\omega$, and $g(\cdot, \cdot) := \tilde{g}(B_1 \cdot, \cdot)$ is a metric compatible with ω. This gives an almost complex structure $I := \omega^{-1} g$, equal to the standard one on $B_{r-\varepsilon} \subset M$ because $B_1 = \operatorname{Id}$ on $B_{r-\varepsilon}$. ■

Symplectic capacity and the pseudoholomorphic curves

THEOREM 2: Let $M = \mathbb{C}P^1 \times T^{2n}$ be the product of $\mathbb{C}P^1$ and a torus, equipped with the standard symplectic structure, and J a compatible almost complex structure. **Then for any $x \in M$ there exists a pseudoholomorphic curve S homologous to $\mathbb{C}P^1 \times \{m\}$ and passing through x.**

This theorem implies Gromov's non-squeezing theorem.

THEOREM: (Gromov) Symplectic capacity of a symplectic cylinder Cyl_1 is equal to π.

Scheme of the proof: We map $\mathrm{Cyl}_1 = B_1^2 \times \mathbb{R}^{2n-2}$ to $M = \mathbb{C}P^1 \times T^{2n-2}$ in such a way that the disk B_1^2 is bijectively mapped to $\mathbb{C}P^1 \backslash \infty$, and $\mathbb{R}^{2n-2}$ is mapped to $\mathbb{R}^{2n-2}/\mathbb{Z}^{2n-2} = T^{2n-2}$. Then we show that the volume of an intersection of a symplectic ball $B_r \subset \mathrm{Cyl}_1$ and a pseudoholomorphic curve passing through $0 \in B_r$ is $\geqslant \pi r^2$, which is impossible when $r > 1$ and the curve is obtained from the Gromov's family obtained in Theorem 2, because all these curves have volume π.

Proof of Gromov's theorem

THEOREM: (Gromov)
Symplectic capacity of a symplectic cylinder Cyl_1 is equal to π.

Proof. Step 1: Let $f_1: B_r \longrightarrow \mathrm{Cyl}_1$ be a symplectic embedding, $r > 1$, and I the usual (flat) almost complex structure on $B_r \subset \mathbb{C}^{n+1}$. Consider the manifold $M = \mathbb{C}P^1 \times T^{2n}$, equipped with the standard symplectic structure, and let $f_2: \mathrm{Cyl}_1 \longrightarrow \mathbb{C}P^1 \times T^{2n}$ be a symplectic map taking $\mathrm{Cyl}_1 = \Delta \times \mathbb{R}^{2n}$ to $\mathbb{C}P^1 \times T^{2n}$ applying the $\mathbb{Z}^{2n}$ quotient on the second argument and the natural symplectomorphism $\Delta \xrightarrow{\sim} \mathbb{C}P^1 \backslash \infty$ on the first argument.

Step 2: Choose the lattice $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n}$ in such a way that its fundamental domain contains $f_1(B_r)$. Then the composition $f_1 \circ f_2$ gives a symplectic embedding $B_r \longrightarrow M = \mathbb{C}P^1 \times T^{2n}$.

We obtained that **Gromov's non-squeezing theorem is deduced from the following result.**

Proof of Gromov's theorem (2)

THEOREM: Let $M = \mathbb{C}P^1 \times T^{2n}$ be equipped with the standard symplectic form, with the symplectic volume of $\mathbb{C}P^1$ equal to π, and $\varphi: \ B_r \longrightarrow M$ a symplectic embedding. **Then $r \leqslant 1$.**

Proof. Step 1: Choose a flat complex structure and the flat Hermitian metric on B_r. Denote by g_0 the corresponding Hermitian metric on $\varphi(B_r)$. Then g_0 can be extended to a Riemannian metric g_1 on M such that $g_0 = g_1$ in a ball $\varphi(B_{r-\varepsilon})$, for some ε such that $r - \varepsilon > 1$. Lemma 1 gives a metric g compatible with the symplectic structure on M and coinciding with g_0 in a ball $\varphi(B_{r-\varepsilon})$. **Replacing B_r by $B_{r-\varepsilon}$, we can add to the assumptions of the theorem the following assumption.**

There exists a compatible almost complex structure such that uts restriction to $\varphi(B_r)$ is equal to the standard complex structure on $B_r \subset \mathbb{C}^{n+1}$.

Proof of Gromov's theorem (3)

THEOREM: Let $M = \mathbb{C}P^1 \times T^{2n}$ equipped with the standard symplectic form, with the symplectic volume of $\mathbb{C}P^1$ equal to π, and $\varphi : B_r \longrightarrow M$ a symplectic embedding. **Assume that there exists a compatible almost complex structure such that its restriction to $\varphi(B_r)$ is equal to the standard complex structure on $B_r \subset \mathbb{C}^{n+1}$. Then $r \leqslant 1$.**

Step 2: Let $x \in M$ be the image of the center of B_r, and $S \subset M$ the pseudo-holomorphic curve which passes through x by Theorem 2. Then $\pi = \int_S \omega_M \geqslant \int_{\varphi^{-1}(S)} \omega$, where ω_M is the symplectic form on M, and ω the symplectic form on B_r. Since S is pseudo-holomorphic, $\int_{\varphi^{-1}(S)} \omega_{B_r}$ **is the Riemannian volume of its intersection with $\varphi(B_r)$.**

Step 3: We obtained a complex curve $D := \varphi^{-1}(S)$ passing through 0 in a ball B_r with flat Riemannian metric and the standard complex structure, with the Riemannian volume $\mathrm{Vol}(D) \leqslant \pi$. Applying the homothety, we obtain a properly embedded complex disk in the ball B of radius 1, passing through 0 and with area $r^{-1}\pi$. **For any $r > 1$, this is impossible, as follows from the following statement,** proven later today.

PROPOSITION 1: Let $D \subset B_1$ be a closed complex disk in a unit ball $B_1 \subset \mathbb{C}^n$, with $0 \in D$. **Then $\mathrm{Vol}(D) \geqslant \pi$.**

Monotonicity formula (1)

PROPOSITION: Let $D \subset B_1$ be a complex curve (that is, a closed 1-dimensional complex subvariety) in a unit ball $B_1 \subset \mathbb{C}^n$, with $0 \in D$. **Then** $\operatorname{Vol}(D) \geqslant \pi$**.**

We deduce it from the **monotonicity formula.**

LEMMA: (monotonicity formula)
Let (B, ω) be the unit ball $B_1(0) \subset \mathbb{R}^{2n}$ equipped with the standard symplectic form ω and the almost complex structure J, and $\Delta \subset B$ a J-holomorphic Riemann surface submanifold passing through 0. We assume that Δ is immersed to B with its boundary in such a way that $\partial\Delta$ is mapped to ∂B. **Let Δ_t be the intersection of Δ with a ball $B_t(0)$ of radius t. Then the function $t \longrightarrow t^{-2} \operatorname{Vol}(\Delta_t)$ is non-decreasing.**

Clearly, $\lim\limits_{t\to 0} \operatorname{Vol}(\Delta_t) = \pi t^2$ (on a very small scale, any smooth disk is flat). **Then the monotonicity lemma gives $\operatorname{Vol}\Delta \geqslant \pi$.** This proves Proposition 1.

Monotonicity formula (2)

LEMMA: (monotonicity formula)
Let (B,ω) be the unit ball $B_1(0) \subset \mathbb{R}^{2n}$ equipped with the standard symplectic form ω and the almost complex structure J, and $\Delta \subset B$ a J-holomorphic Riemann surface submanifold passing through 0. We assume that Δ is immersed to B with its boundary in such a way that $\partial\Delta$ is mapped to ∂B. **Let Δ_t be the intersection of Δ with a ball $B_t(0)$ of radius t. Then the function $t \longrightarrow t^{-2}\operatorname{Vol}(\Delta_t)$ is non-decreasing.**

Proof. Step 1: Let $C(\partial\Delta) \subset B$ be the 3-dimensional cone obtained as a union of all intervals connecting 0 and points of $\partial\Delta \subset \partial B$. Then

$$\operatorname{Vol} C(\partial\Delta) \geqslant \int_{C(\partial\Delta)} \omega = \int_\Delta \omega = \operatorname{Vol}\Delta.$$

The first equality follows from the Stokes' theorem, because $\partial\Delta = \partial C(\partial\Delta)$, and the first inequality holds because ω is a calibration.

Step 2: The formula for the cone volume gives $\operatorname{Vol} C(\partial\Delta) = \frac{1}{2} l(\partial\Delta)$, where $l(\partial\Delta)$ is the area of the boundary $\partial\Delta$. **This implies $\operatorname{Vol}\Delta \leqslant \frac{1}{2} l(\partial\Delta)$.**

Monotonicity formula (3)

LEMMA: (monotonicity formula)
Let (B,ω) be the unit ball $B_1(0) \subset \mathbb{R}^{2n}$ equipped with the standard symplectic form ω and the almost complex structure J, $\Delta \subset B$ a J-holomorphic Riemann surface submanifold passing through 0. We assume that Δ is immersed to B with its boundary in such a way that $\partial\Delta$ is mapped to ∂B. **Let Δ_t be the intersection of Δ with a ball $B_t(0)$ of radius t. Then the function $t \longrightarrow t^{-2}\operatorname{Vol}(\Delta_t)$ is non-decreasing.**

Step 2: We have obtained $\operatorname{Vol}\Delta \leqslant \frac{1}{2}l(\partial\Delta)$.

Step 3: The same argument applied to Δ_t gives $t^{-2}\operatorname{Vol}\Delta_t \leqslant \frac{1}{2t}l(\partial\Delta_t)$. (homothety maps the 2-dimensional Riemann volume μ to $t^2\mu$). On the other hand, $\frac{d}{dt}\operatorname{Vol}\Delta_t \geqslant l(\partial\Delta_t)$, because the volume of the strip of Δ between $\partial\Delta_t$ and $\partial\Delta_{t+\varepsilon}$ is bounded by $\varepsilon l(\partial\Delta_{t+\varepsilon})$. This gives

$$\operatorname{Vol}\Delta_t \leqslant \frac{t}{2}l(\partial\Delta_t) \leqslant \frac{t}{2}\frac{d}{dt}\operatorname{Vol}\Delta_t.$$

Step 4: Let $f(t) := \operatorname{Vol}\Delta_t$. The last formula of Step 3 gives $f(t) \leqslant \frac{t}{2}f'(t)$, hence

$$\frac{d}{dt}t^{-2}f(t) = t^{-2}f'(t) - 2t^{-3}f(t) \geqslant t^{-2}f'(t) - t^{-2}f'(t) = 0.$$

■