Symplectic geometry

lecture 12: Foliations and holonomy

Misha Verbitsky

HSE, room 306, 16:20,

October 13, 2021

Frobenius bracket

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let $\Pi: TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that **$\Pi([x, y])$ is $C^\infty(M)$-linear as a function of x and y.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **integrable**, or **holonomic**, or **involutive**, if its Frobenius form vanishes.

Frobenius theorem and foliations

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \stackrel{\pi}{\longrightarrow} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

Holonomy of a foliation with compact leaves

DEFINITION: Let $\mathcal{F}$ be a smooth foliation on M. Suppose that all leaves of $\mathcal{F}$ are compact (in this case $\mathcal{F}$ is called **a foliation with compact leaves**).

Let $F \subset M$ be a compact leaf of $\mathcal{F}$, and U its tubular neighbourhood. Denote by π a smooth retraction of U to F. For U sufficiently small, we may assume that π is locally a diffeomorphism on each leaf of $\mathcal{F}$. Then π restricted to a compact leaf $F_1 \subset F$ is a covering. In particular, every path $\gamma \subset F$ can be lifted to a covering $\gamma_1 \in F_1$. Let $S := \pi^{-1}(x)$, where x is the starting point of a loop $\gamma : [0,1] \longrightarrow F$. Then γ_1 is uniquely determined by $\gamma_1(0)$ and gives a $H_\gamma : S \longrightarrow S$ mapping $\gamma_1(0)$ to $\gamma_1(1)$. **This construction defines a group homomorphism $\pi_1(F) \longrightarrow \operatorname{Diff}_x(S_x)$,** where S_x is a germ of S in x, and $\operatorname{Diff}_x(S_x)$ denotes the group of diffeomorphisms of this germ.

DEFINITION: (Ehresmann)
The homomorphism $\pi_1(F) \longrightarrow \operatorname{Diff}_x(S_x)$ is called **the holonomy of the foliation $\mathcal{F}$ in F.**

REMARK: Holonomy is well defined for any leaf of a foliation, **compactness of its leaves is not necessary.** Moreover, a germ of a foliation in a neighbourhood of a closed leaf **is uniquely (up to a diffeo) determined by its holonomy.**

Holonomy of a foliation with compact leaves

REMARK: Holonomy of a foliation with compact leaves is finite in dimension 3, by a theorem of D. B. A. Epstein. However, in dimension 5 D. Sullivan **produced an S^1-foliation on S^5 with infinite holonomy.**

EXERCISE: Let G be a compact Lie group acting on a manifold M. Prove that **all orbits have dimension** $\dim G$ **if and only if for some basis** $g_1, ...g_n \in$ Lie G **the corresponding vector fields on** M **are linearly independent everywhere.**

DEFINITION: In this case, the action of G on M is called **locally free.**

THEOREM: Let G be a compact Lie group which locally freely acts on a manifold M, and $\mathcal{F}$ the corresponding foliation, with its leaves being the orbits of G. **Then the holonomy of $\mathcal{F}$ is finite.** Moreover, the leaf space M/G locally in a neighbourhood of $[F]$ **is homeomorphic to $\mathbb{R}^n/\Gamma$, where Γ is the holonomy of $\mathcal{F}$ in F.**

Holonomy of a foliation with transitive group action on its leaves

THEOREM: Let G be a compact Lie group with locally freely acts on a manifold M, and $\mathcal{F}$ the corresponding foliation, with its leaves being the orbits of G. **Then the holonomy of $\mathcal{F}$ is finite.** Moreover, the leaf space M/G locally in a neighbourhood of $[F]$ **is homeomorphic to $\mathbb{R}^n/\Gamma$, where Γ is the holonomy of $\mathcal{F}$ in F.**

Proof. Step 1: Choose a G-invariant Riemannian metric on M by taking any Riemannian metric and averaging it with G. Then for any $r \in \mathbb{R}^{>0}$, an r-neighbourhood U of a leaf F is G-invariant. Therefore, **U contains the whole leaf F_1 of $\mathcal{F}$ if it contains a point of F_1.**

Let $\pi: U \longrightarrow F$ a smooth retraction, $S := \pi^{-1}(x)$, and Γ the holonomy of $\mathcal{F}$ in F. Consider a leaf F_1 of $\mathcal{F}$ passing through $x_1 \in S$. **Then $F_1 \cap S = \Gamma \cdot x_1$, hence $U/G = S/\Gamma$.** We proved the second claim of the theorem.

Step 2: To see that Γ is positive, we choose σ inverse to the Riemannian geodesic (exponential) map in the direction orthogonal to F. Consider the map $\mu: S \times G \longrightarrow U$ mapping (s,g) to $g(s)$. This map is by construction surjective and each point $x_1 \in S \cap F_1$ has precisely Γ_{F_1} preimages. Let Γ_F be the subgroup of G fixing $x \in F$. Then $\mu(\Gamma_F)$ maps $S \subset U$ to itself, and this action coincides with Γ. Since G is compact, Γ_F is finite. ■