Symplectic geometry

lecture 13: symplectic reduction

Misha Verbitsky

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Frobenius theorem and foliations (reminder)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then $[B, B] \subset B$ if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

Holonomy of a foliation with compact leaves (reminder)

DEFINITION: Let $\mathcal{F}$ be a smooth foliation on M. Suppose that all leaves of $\mathcal{F}$ are compact (in this case $\mathcal{F}$ is called **a foliation with compact leaves**).

Let $F \subset M$ be a compact leaf of $\mathcal{F}$, and U its tubular neighbourhood. Denote by π a smooth retraction of U to F. For U sufficiently small, we may assume that π is locally a diffeomorphism on each leaf of $\mathcal{F}$. Then π restricted to a compact leaf $F_1 \subset F$ is a covering. In particular, every path $\gamma \subset F$ can be lifted to a covering $\gamma_1 \in F_1$. Let $S := \pi^{-1}(x)$, where x is the starting point of a loop $\gamma: [0,1] \longrightarrow F$. Then γ_1 is uniquely determined by $\gamma_1(0)$ and gives a $H_\gamma: S \longrightarrow S$ mapping $\gamma_1(0)$ to $\gamma_1(1)$. **This construction defines a group homomorphism $\pi_1(F) \longrightarrow \operatorname{Diff}_x(S_x)$,** where S_x is a germ of S in x, and $\operatorname{Diff}_x(S_x)$ denotes the group of diffeomorphisms of this germ.

DEFINITION: (Ehresmann)
The homomorphism $\pi_1(F) \longrightarrow \operatorname{Diff}_x(S_x)$ is called **the holonomy of the foliation $\mathcal{F}$ in F.**

REMARK: Holonomy is well defined for any leaf of a foliation, **compactness of its leaves is not necessary.** Moreover, a germ of a foliation in a neighbourhood of a closed leaf **is uniquely (up to a diffeo) determined by its holonomy.**

Holonomy of a foliation with compact leaves (reminder)

REMARK: Holonomy of a foliation with compact leaves is finite in dimension 3, by a theorem of D. B. A. Epstein. However, in dimension 5 D. Sullivan **produced an S^1-foliation on S^5 with infinite holonomy.**

EXERCISE: Let G be a compact Lie group acting on a manifold M. Prove that **all orbits have dimension** $\dim G$ **if and only if for some basis** $g_1, ...g_n \in$ Lie G **the corresponding vector fields on M are linearly independent everywhere.**

DEFINITION: In this case, the action of G on M is called **locally free.**

THEOREM: Let G be a compact Lie group which locally freely acts on a manifold M, and $\mathcal{F}$ the corresponding foliation, with its leaves being the orbits of G. **Then the holonomy of $\mathcal{F}$ is finite.** Moreover, the leaf space M/G locally in a neighbourhood of $[F]$ **is homeomorphic to $\mathbb{R}^n/\Gamma$, where Γ is the holonomy of $\mathcal{F}$ in F.**

Holonomy of a foliation with transitive group action on its leaves (reminder)

THEOREM: Let G be a compact Lie group with locally freely acts on a manifold M, and $\mathcal{F}$ the corresponding foliation, with its leaves being the orbits of G. **Then the holonomy of $\mathcal{F}$ is finite.** Moreover, the leaf space M/G locally in a neighbourhood of $[F]$ **is homeomorphic to $\mathbb{R}^n/\Gamma$, where Γ is the holonomy of $\mathcal{F}$ in F.**

Proof. Step 1: Choose a G-invariant Riemannian metric on M by taking any Riemannian metric and averaging it with G. Then for any $r \in \mathbb{R}^{>0}$, an r-neighbourhood U of a leaf F is G-invariant. Therefore, **U contains the whole leaf F_1 of $\mathcal{F}$ if it contains a point of F_1.**

Let $\pi: U \longrightarrow F$ a smooth retraction, $S := \pi^{-1}(x)$, and Γ the holonomy of $\mathcal{F}$ in F. Consider a leaf F_1 of $\mathcal{F}$ passing through $x_1 \in S$. **Then $F_1 \cap S = \Gamma \cdot x_1$, hence $U/G = S/\Gamma$.** We proved the second claim of the theorem.

Step 2: To see that Γ is positive, we choose σ inverse to the Riemannian geodesic (exponential) map in the direction orthogonal to F. Consider the map $\mu: S \times G \longrightarrow U$ mapping (s, g) to $g(s)$. This map is by construction surjective and each point $x_1 \in S \cap F_1$ has precisely Γ_{F_1} preimages. Let Γ_F be the subgroup of G fixing $x \in F$. Then $\mu(\Gamma_F)$ maps $S \subset U$ to itself, and this action coincides with Γ. Since G is compact, Γ_F is finite. ■

Orbifolds

DEFINITION: We say that a topological space **has quotient singularities** if it is locally homeomorphic to $\mathbb{R}^n/\Gamma$, where Γ is a finite group acting on $\mathbb{R}^n$ by diffeomorphisms.

It turns out that all "analysis on manifolds" can be performed on such varieties, if we define **the orbifolds** (but it would take time). In particular,

THEOREM: Let M be a manifold equipped with a locally free action of a Lie group. **Then M/G is an orbifold.**

Basic forms

DEFINITION: Let $B \subset M$ be an involutive sub-bundle, tangent to a foliation $\mathcal{F}$ on M, and $U \xrightarrow{\pi_U} U/\mathcal{F}$ its leaf space, defined for a sufficiently small $U \subset M$. **A basic form** on $(M, \mathcal{F})$ is a form $\alpha \in \Lambda^k(M)$ such that for each local leaf space $U \xrightarrow{\pi_U} U/\mathcal{F}$, one can represent α as a pullback, $\alpha = \pi_U^*(\alpha_0)$ for a form α_0 on $U/\mathcal{F}$.

PROPOSITION: A form $\alpha \in \Lambda^k(M)$ **is basic with respect to $\mathcal{F}$ if and only if for any vector field X tangent to $\mathcal{F}$, one has $i_X(\alpha) = 0$ and $\operatorname{Lie}_X(\alpha) = 0$.**

Proof: Let $x_1, ..., x_n, y_1, ..., y_m$ be a coordinate system on U such that π_U maps $(x_1, ..., x_n, y_1, ..., y_m)$ to $(x_1, ..., x_n)$. Then a form α is expressed through a sum of coordinate monomials and functions as $\alpha = \sum f_I \alpha_I$ is basic if and only if the functions f_I are independent from y_i and the coordinate monomials α_I do not contain dy_i. ■

COROLLARY: Let α be a closed differential form on a manifold M, equipped with a foliation $\mathcal{F}$. **Then α is basic if and only if $i_X(\alpha) = 0$ for any vector field X tangent to $\mathcal{F}$.**

Proof: Cartan's formula gives $\operatorname{Lie}_X \alpha = i_X(d\alpha) + d(i_X\alpha)$; when α is closed, this is equivalent to $\operatorname{Lie}_X \alpha = d(i_X\alpha)$. ■

Characteristic foliation

DEFINITION: Let (M,ω) be a symplectic manifold. A submanifold $Z\subset M$ is called **coisotropic** if $\dim_{\mathbb R} Z \geqslant \dim_{\mathbb R} M$ and $\omega|_Z$ has rank $\dim_{\mathbb R} M - \dim_{\mathbb R} Z$ (minimal possible), or, equivalently, $(TZ)^{\perp_\omega}\subset TZ$, where $(TZ)^{\perp_\omega} := \{x\in TZ \ \mid\ i_x\omega = 0\}$

DEFINITION: Let $Z\subset (M,\omega)$ be a coisotropic submanifold. The bundle $(TZ)^{\perp_\omega}$ is called **the characteristic bundle of Z**.

THEOREM: Let $Z\subset (M,\omega)$ be a coisotropic submanifold, and $K\subset TM$ its characteristic bundle. **Then $[K,K]\subset K$, hence K is tangent to a foliation** $\mathcal F$ which is called **the characteristic foliation of Z.** Moreover, **the restriction $\omega|_Z$ is basic, and symplectic on the leaf space of the characteristic foliation.**

Characteristic foliation (2)

THEOREM: Let $Z \subset (M,\omega)$ be a coisotropic submanifold, and $K \subset TM$ its characteristic bundle. **Then $[K,K] \subset K$, hence K is tangent to a foliation** $\mathcal{F}$ which is called **the characteristic foliation of Z.** Moreover, **the restriction $\omega|_Z$ is basic, and symplectic on the leaf space of the characteristic foliation.**

Proof: To see that $[K,K] \subset K$, we use Cartan's formula for de Rham differential

$$0 = d\omega(X,Y,Z) = \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) + \\ + \operatorname{Lie}_X \omega(Y,Z) + \operatorname{Lie}_Y \omega(Z,X) + \operatorname{Lie}_Z \omega(X,Y).$$

If $X,Y \in K$, all terms in this sum vanish, except, possibly,

$$0 = d\omega(X,Y,Z) = \omega([X,Y],Z),$$

hence $[X,Y] \in K$.

Now, **ω is K-basic because it is closed and vanishes on K,** and symplectic on the leaf space $M/\mathcal{F}$ because it has rank $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} Z$, and $\dim M/\mathcal{F} = \dim_{\mathbb{R}} M - \dim_{\mathbb{R}} Z$. ■

Cartan's formula and symplectomorphisms (reminder)

We denote the Lie derivative along a vector field as $\operatorname{Lie}_x : \Lambda^i M \longrightarrow \Lambda^i M$, and contraction with a vector field by $i_x : \Lambda^i M \longrightarrow \Lambda^{i-1} M$.

Cartan's formula: $d \circ i_x + i_x \circ d = \operatorname{Lie}_x$.

REMARK: Let (M, ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms, and $\mathfrak{g}$ its Lie algebra. For any $g \in \mathfrak{g}$, denote by ρ_g the corresponding vector field. Then $\operatorname{Lie}_{\rho_g} \omega = 0$, giving $d(i_{\rho_g}(\omega)) = 0$. **We obtain that $i_{\rho_g}(\omega)$ is closed, for any $g \in \mathfrak{g}$.**

DEFINITION: A Hamiltonian of $g \in \mathfrak{g}$ is a function h on M such that $dh = i_{\rho_g}(\omega)$.

Moment maps

DEFINITION: (M,ω) be a symplectic manifold, and G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**, that is, up to addition of a **central** (that is, G-invariant) vector $c \in \mathfrak{g}^*$.

CLAIM: An equivariant moment map exists whenever $H^1(G,\mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists. Further on, all moment maps will be tacitly considered equivariant.

Weinstein-Marsden theorem

DEFINITION: (Weinstein-Marsden) (M,ω) be a symplectic manifold, G a compact Lie group freely acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

Proof. Step 1: $T_x(\mu^{-1}(c)) = d\mu^{-1}(0)$. However, the space $\langle d\mu, g\rangle \subset \Lambda^1 M$ is ω-dual to the space $\tau(\mathfrak{g})$ of vector fields tangent to the G-action, **hence $d\mu^{-1}(c) = \tau(\mathfrak{g})^\perp$.**

Step 2: Since μ is G-equivariant, G preserves $\mu^{-1}(c)$, hence $\tau(\mathfrak{g}) \subset d\mu^{-1}(0)$. This implies that $\tau(\mathfrak{g}) \subset TM$ **is isotropic** (that is, $\omega\big|_{\tau(\mathfrak{g})} = 0$). Its ω-orthogonal complement in T_xM is $T_x(\mu^{-1}(c))$ (Step 1).

Step 3: Consider the **characteristic foliation** $\mathcal{F}$ on $\mu^{-1}(c)$. It is a bundle because $\mu^{-1}(c) \subset M$ is coisotropic. From Step 2 **we obtain that $\mathcal{F} = \tau(\mathfrak{g})$.**

Step 4: Since $\omega\big|_{\mu^{-1}(c)}$ is closed, it satisfies $\operatorname{Lie}_v(\omega) = 0$ for all $v \in \mathcal{F}$. This implies that it is *basic*, that is, lifted from the leaf space of characteristic foliation, identified with $M/\!\!/G$. ■