Symplectic geometry

lecture 15: Geometric invariant theory

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Moment maps (reminder)

DEFINITION: (M,ω) be a symplectic manifold, and G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**, that is, up to addition of a central vector $c \in \mathfrak{g}^*$.

CLAIM: An equivariant moment map exists whenever $H^1(G, \mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists. Further on, all moment maps will be tacitly considered equivariant.

Weinstein-Marsden theorem (reminder)

DEFINITION: (Weinstein-Marsden) (M,ω) be a symplectic manifold, G a compact Lie group freely acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

Proof. Step 1: $T_x(\mu^{-1}(c)) = d\mu^{-1}(0)$. However, the space $\langle d\mu, g\rangle \subset \Lambda^1 M$ is ω-dual to the space $\tau(\mathfrak{g})$ of vector fields tangent to the G-action, **hence $d\mu^{-1}(c) = \tau(\mathfrak{g})^\perp$.**

Step 2: Since μ is G-equivariant, G preserves $\mu^{-1}(c)$, hence $\tau(\mathfrak{g}) \subset d\mu^{-1}(0)$. This implies that **$\tau(\mathfrak{g}) \subset TM$ is isotropic** (that is, $\omega\big|_{\tau(\mathfrak{g})} = 0$). Its ω-orthogonal complement in T_xM is $T_x(\mu^{-1}(c))$ (Step 1).

Step 3: Consider the **characteristic foliation** $\mathcal{F}$ on $\mu^{-1}(c)$. It is a bundle because $\mu^{-1}(c) \subset M$ is coisotropic. From Step 2 **we obtain that $\mathcal{F} = \tau(\mathfrak{g})$.**

Step 4: Since $\omega\big|_{\mu^{-1}(c)}$ is closed, it satisfies $\operatorname{Lie}_v(\omega) = 0$ for all $v \in \mathcal{F}$. This implies that it is *basic*, that is, lifted from the leaf space of characteristic foliation, identified with $M/\!\!/G$. ■

Holomorphic vector fields

DEFINITION: A holomorphic vector field is a vector field $X \in TM$ satisfying $\operatorname{Lie}_X I = 0$, that is, such that the corresponding diffeomorphism flow e^{tX} is holomorphic.

REMARK: Let (M, I) be a compact complex manifold. Then the group of biholomorphisms of M is a Lie group whose Lie algebra is the space of (real) holomorphic vector fields.

REMARK: It is not hard to see that **$I(X)$ is holomorphic whenever X is holomorphic** (**prove this**). Then $X \longrightarrow I(X)$ **defines the complex structure on the Lie algebra of holomorphic vector fields.**

Symplectic reduction and a Kähler potential (reminder)

DEFINITION: Let $d^c := IdI^{-1}$. **Kähler potential** on a Kähler manifold (M,ω) is a function ψ such that $dd^c\psi = \omega$.

PROPOSITION: Let G be a real Lie group acting on a Kähler manifold M by holomorphic isometries, and ψ a G-invariant Kähler potential. **Then the moment map $\mathfrak{g} \times M \xrightarrow{\mu_g} \mathbb{R}$ can be written as $g, m \longrightarrow \operatorname{Lie}_{Iv}\psi$,** where $v = \tau(g) \in TM$ is the tangent vector field associated with $g \in \mathfrak{g}$.

Proof: Since ψ is G-invariant, and I is G-invariant, we have $0 = \operatorname{Lie}_v d^c\psi = i_v(dd^c\psi) + d(i_v d^c\psi)$. Using $\omega = dd^c\psi$, we rewrite this equation as $i_v\omega = -d(\langle d^c\psi, v\rangle)$, giving an equation for the moment map $\mu_g = -\langle d^c\psi, v\rangle$. Acting by I on both sides, we obtain $\mu_g = \langle d\psi, Iv\rangle = \operatorname{Lie}_{Iv}\psi$. ■

COROLLARY: Let V be a Hermitian representation of a compact Lie group G. **Then the corresponding moment map can be written as $\mu_g(v) = \operatorname{Lie}_{Ig}|v|^2 = \frac{1}{4}\langle v, Ig(v)\rangle$.** ■

Moment map and extrema on 1-parametric orbit

REMARK: Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure h, and $G_{\mathbb{C}} \subset GL(n,\mathbb{C})$ its complexification. Denote by $\mu: V \longrightarrow \mathfrak{g}^*$ the moment map. Since $\mu_g(v) = \operatorname{Lie}_{Ig} |v|^2$, a vector $z \in V$ **belongs to $\mu^{-1}(0)$ if and only if the function $l: G_{\mathbb{C}} \cdot z \longrightarrow \mathbb{R}$, $l(z) = |z|^2$ on the orbit $G_{\mathbb{C}}$ has extremum in z.** It turns own that $z \longrightarrow l(z)$ **is convex**, in the following sense.

CLAIM: Let $V = \mathbb{C}^n$, $A \in \mathfrak{u}(V)$ be an anti-Hermitian endomorphism, and $G_A := e^{tA} \subset GL(V)$, $t \in \mathbb{C}$ the corresponding 1-parametric subgroup. **Then $\frac{d^2}{du^2}|e^{(t+u)A}(z)|^2 = 4|A(e^{tA}(z))|^2$,** when $u \in \sqrt{-1}\,\mathbb{R}$.

Proof: Let A be diagonalized in an orthonormal basis $x_1, ..., x_n \in \mathbb{R}$, such that $A(x_i) = \sqrt{-1}\, w_i x_i$, $w_i \in \mathbb{R}$, and h the Hermitian form. The operator $e^{tA}(x_i) = e^{\sqrt{-1}\, t w_i} x_i$ is an isometry when t is real, and is Hermitian self-adjoint when t is imaginary. This gives $\frac{d}{du}|e^{(t+u)A}(z)|^2 = 2h(Ae^{(t)A}(z), e^{(t)A}(z))$. Taking the derivative in u once again, we obtain $\frac{d^2}{du^2}|e^{(t+u)A}(z)|^2 = 2h(Ae^{(t)A}(z), Ae^{(t)A}(z))$.
■

COROLLARY: The function l **is convex on G_A,** and **strictly convex in imaginary direction,** unless $A = 0$. ■

Moment map and extrema on 1-parametric orbit

COROLLARY: Let A be diagonalized in an orthonormal basis $x_1, ..., x_n \in \mathbb{R}$, such that $A(x_i) = \sqrt{-1}\, w_i x_i$, $w_i \in \mathbb{R}$, and $z = \sum \alpha_i x_i$, with all $\alpha_i \neq 0$. Then the function l **has a minimum on the line $e^{\sqrt{-1}\,\mathbb{R}A}(z)$ if and only if there are two basis vectors x_l, x_k such that $w_l < 0$ and $w_k > 0$.** Moreover, l **has no other extrema on $e^{\sqrt{-1}\,\mathbb{R}A}(z)$, unless $A = 0$.**

Proof: The function $u \longrightarrow |e^{(t+u)A}(z)|^2$ is strictly convex along the imaginary axis, and constant along the real axis. Therefore, it has at most one extremum, and it is the minimum. It has minimum if and only if $\lim |e^{(t+u)A}(z)|^2 = \infty$ as $t \longrightarrow \pm\infty$, which happens if and only if $w_l < 0$ and $w_k > 0$ for some k, l. ■

Extrema of the length function on $G_{\mathbb{C}}$-orbits

THEOREM: Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure h, and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ its complexification. Let $\mathfrak{g}_{\mathbb{C}} = \operatorname{Lie}(G_{\mathbb{C}}) = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Consider the function $\varphi: \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{R}$ taking g to $l(e^g z)$. **Then φ is constant in real direction, convex in imaginary direction, and satisfies $\frac{d^2}{du^2}|\varphi(u)(z)|^2 = 4h(u(z), u(z))$ where $u \in \operatorname{im} \mathfrak{g}_{\mathbb{C}}$.**

Proof: Write $u \in I(\mathfrak{u}(n))$ in an appropriate orthonormal basis $x_1, ..., x_n$ as $u(x_i) = w_i x_i$, $w_i \in \mathbb{R}$. Let $z = \sum \alpha_i x_i$. Then $\frac{d}{du}|\varphi(u)(z)|^2 = 2\sum w_i |x_i|^2 = 2h(u(z), z)$, and $\frac{d^2}{du^2}|\varphi(u)(z)|^2 = 4h(u(z), u(z))$. ■

COROLLARY 1: Either the function l **has no extremal points on the orbit $G_{\mathbb{C}} \cdot z$, or l has a minimum.** In the second case, **the set of points on $G_{\mathbb{C}}$ where l has a minimum is an orbit of G.** ■

Stable and unstable orbits

DEFINITION: Let $G \subset U(n)$ be a Lie group acting on a complex vector space $V = \mathbb{C}^n$, equipped with the standard Hermitian structure, and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ its complexification. An orbit $G_{\mathbb{C}} \cdot z$, $z \neq 0$ is called **stable** if l reaches minimum on $G_{\mathbb{C}} \cdot z$, **unstable** if 0 belongs to the closure of $G_{\mathbb{C}} \cdot z$, and **(strictly) semistable** if if it is not stable and not unstable.

THEOREM: Let $G \subset U(V)$ be a group acting on a complex Hermitian vector space $V = \mathbb{C}^n$, and $z \in V \backslash 0$. **Then an orbit $G_{\mathbb{C}} \cdot z$ is stable if and only if it intersects the zero set of the moment map $\mu(v, z) = \operatorname{Lie}_{Iv}(l)(z)$.** Moreover, **$G$ acts on $G_{\mathbb{C}} \cdot z \cap \mu^{-1}(0)$ transitively.**

Proof: Extrema of l on $G_{\mathbb{C}} \cdot z$ are its minima because l is convex. The extrema of l are zero set of μ because $\mu(v, z) = \operatorname{Lie}_{Iv}(l)(z)$. The set of extrema is one G-orbit by Corollary 1. ■

Hilbert-Mumford criterion of stability

EXERCISE: Let $\rho: U(1) \longrightarrow \operatorname{End}(V)$, $V = \mathbb{C}^n$ be a complex Hermitian representation of $U(1)$. Then **there exists an orthonormal basis $x_1, ..., x_n$ in V such that $g(x_i) = \sqrt{-1}\, 2\pi w_i x_i$,** where $w_i \in \mathbb{Z}$ are integer numbers called **the weights** of the action.

Proof: Since the action is unitary, $\rho(t)$ is diagonalizable. The numbers w_i are integer because $\rho\Big|_{\sqrt{-1}\,\mathbb{R}}$ factorizes through $U(1)$. ■

THEOREM: Let $G = U(1)$ act on a complex Hermitian vector space $(V = \mathbb{C}^n, h)$, $G_{\mathbb{C}} = \mathbb{C}^*$ the corresponding complex Lie group, and $z \in V$ a non-zero vector, $z = \sum \alpha_{l_k} x_{l_k}$, where α_{l_k} are all non-zero. Consider the weight decomposition of the generator of this action: $A(x_i) = \sqrt{-1}\, 2\pi w_i x_i$. Then

- **$G_{\mathbb{C}} \cdot z$ is unstable if and only if all w_{l_k} are positive or negative,**

- **$G_{\mathbb{C}} \cdot z$ is stable if and only if some w_{l_k} are positive while others are negative.**

• $G_{\mathbb{C}} \cdot z$ **is strictly semistable if some w_{l_k} vanish and all others are positive or negative.**

Proof: If some w_{l_k} are positive while others are negative, one has $\lim_{\operatorname{im} t\to\pm\infty} |e^{t\sqrt{-1}\,A}(z)| = \infty$, hence l reaches the minimum somewhere on the imaginary axis.

If all weights are positive or negative, one has $\lim_{\operatorname{im} t\to\infty} |e^{t\sqrt{-1}\,A}(z)| = \infty$ and $\lim_{\operatorname{im} t\to-\infty} e^{t\sqrt{-1}\,A}(z) = 0$ or vice versa, and the orbit is unstable.

If all weights are $\geqslant 0$ or $\leqslant 0$, with one of the weights equal to 0, one has $\lim_{\operatorname{im} t\to\infty} |e^{t\sqrt{-1}\,A}(z)| = \infty$ and $\lim_{\operatorname{im} t\to-\infty} e^{t\sqrt{-1}\,A}(z) = z$, and the orbit is strictly semistable. ■

Set of stable orbits

For $G_{\mathbb{C}} = \mathbb{C}^*$, the following theorem immediately follows from the Hilbert-Mumford criterion.

PROPOSITION: Let $G \subset U(n)$ be a Lie group acting on a complex Hermitian vector space $(V = \mathbb{C}^n, h)$, and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ its complexification, and V_s the set of all $z \in V$ such that the orbit $G_{\mathbb{C}} \cdot z$ is stable. **Then $V_s \subset V$ is open.**

Proof. Step 1: Let $\overline{B}_R \subset V$ be a closed ball of radius R. The orbit $z \in V_s$ is stable if and only if for each $R \in \mathbb{R}^{>0}$, the intersection $\overline{B}_R \cap G_{\mathbb{C}} \cdot z$ is compact for all $R \in \mathbb{R}^{>0}$. Indeed, if it is compact, l reaches minimum somewhere on $G_{\mathbb{C}} \cdot z$. On the other hand, of l reaches its minimum, one has $\lim_{\operatorname{im} t \to \pm\infty} |e^{t\sqrt{-1}\, A}(z)| = \infty$ for all $\sqrt{-1}\, A \in \sqrt{-1}\, \mathfrak{g}$, hence $\overline{B}_R \cap G_{\mathbb{C}} \cdot z$ is compact.

Step 2: If $\lim_{\operatorname{im} t \to \pm\infty} |e^{t\sqrt{-1}\, A}(z)| = \infty$ for all $\sqrt{-1}\, A \in \sqrt{-1}\, \mathfrak{g}$, one has $\lim_{\operatorname{im} t \to \pm\infty} |e^{t\sqrt{-1}\, A}(z+u)| = \infty$ for u sufficiently small, hence this condition is open in $z \in V$. ■

Geometric Invariant Theory

REMARK: The following theorem is identifies "the GIT reduction" (taking a $G_{\mathbb{C}}$-quotient of the union of all stable orbits) and the symplectic reduction.

THEOREM: Let $G\subset U(n)$ be a Lie group acting on a complex Hermitian vector space $(V=\mathbb{C}^n, h)$, and $G_{\mathbb{C}}\subset GL(n,\mathbb{C})$ its complexification. Denote by $\mu: V \longrightarrow \mathfrak{g}^*$ the moment map, $\mu(g,z) := (\operatorname{Lie}_{Ig} l)(z)$. **Then an orbit $G_{\mathbb{C}}\cdot z$ is stable if and only if $G_{\mathbb{C}}\cdot z \cap \mu^{-1}(0) \neq 0$.** Moreover, $G_{\mathbb{C}}\cdot z\cap\mu^{-1}(0)$ **is precisely one G-orbit, and $\mu^{-1}(0)/G = V_s/G_{\mathbb{C}}$,** where $V_s\subset V$ is the union of all stable orbits.

Proof: $G_{\mathbb{C}}\cdot z$ is stable if and only if $G_{\mathbb{C}}\cdot z\cap\mu^{-1}(0)\neq 0$ because $\mu^{-1}(0)$ intersects the orbit in the points where the length l is minimal. This intersection set is precisely one Gorbit, which gives $\mu^{-1}(0)/G = V_s/G_{\mathbb{C}}$. ∎