Trisymplectic manifolds

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Complexification of a manifold

DEFINITION: Let M be a complex manifold, equipped with an anticomplex involution ι. The fixed point set $M_{\mathbb{R}}$ of ι is called **a real analytic manifold**, and a germ of M in $M_{\mathbb{R}}$ is called **a complexification** of $M_{\mathbb{R}}$.

QUESTION: **What is a complexification of a Kähler manifold** (considered as real analytic variety)?

THEOREM: (D. Kaledin, B. Feix) Let M be a real analytic Kähler manifold, and $M_{\mathbb{C}}$ its complexification. **Then $M_{\mathbb{C}}$ admits a hyperkähler structure,** determined uniquely and functorially by the Kähler structure on M.

QUESTION: **What is a complexification of a hyperkähler manifold?**

THIS IS THE MAIN SUBJECT OF TODAY'S TALK.

(A joint work with Marcos Jardim).

Plan of the talk:

1. Trisymplectic structures on a vector space (linear algebra).

2. Trisymplectic structures on a manifold (differential geometry).

3. Trisymplectic structure on the space of rational lines in the twistor space (hyperkähler geometry).

4. Applications to the instanton spaces.

Trisymplectic structure on a vector space

DEFINITION: A **trisymplectic structure** on a complex vector space of dimension $2n$ is a 3-dimensional space $\mathbf{\Omega} \subset \Lambda^2 V$ of complex linear 2-forms, such that any $\eta \in \mathbf{\Omega}$ has rank $2n$, n or 0.

REMARK: It is easy to see that **Ω contains a symplectic form.**

PROPOSITION: Given two symplectic forms $\omega_1, \omega_2 \in \mathbf{\Omega}$, consider the map $\varphi_{\Omega_1,\Omega_2} := \omega_1 \circ \omega_2^{-1} \in \operatorname{End}(V)$. Then $\varphi_{\Omega_1,\Omega_2}$ can be expressed in an appropriate basis by the matrix

$$\varphi_{\omega_1,\omega_2} = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda' & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda' \end{pmatrix},$$

with the eigenspaces of equal dimension.

THEOREM: Let $(V, \mathbf{\Omega})$ be a be a trisymplectic vector space, and $H \subset \operatorname{End}(V)$ the algebra generated by $\varphi_{\Omega_1,\Omega_2}$, for all $\omega_1, \omega_2 \in \mathbf{\Omega}$. **Then H is isomorphic to the matrix algebra $\operatorname{Mat}(2)$,** acting on V in a standard way.

Trisymplectic structures as Mat(2)-representations

DEFINITION: Let V be a complex vector space with the **standard action** of the matrix algebra Mat(2), i.e. $V \cong V_0 \otimes \mathbb{C}^2$ and Mat(2) acts only through the second factor.

CLAIM: Consider the natural $SL(2)$-action on V induced by Mat(2), and extend it multiplicatively to all tensor powers of V. Let $g \in \operatorname{Sym}^2_{\mathbb{C}}(V)$ be an $SL(2)$-invariant, non-degenerate 2-form on V, and $\{I, J, K\}$ a quaternionic basis in Mat(2) Then

$$g(x, Iy) = g(Ix, I^2 y) = -g(Ix, y)$$

hence the form $\Omega_I(\cdot,\cdot) := g(\cdot, I\cdot)$ is a symplectic form, obviously non-degenerate; the forms Ω_J, Ω_K have the same properties. Let $\mathbf{\Omega} := \langle \Omega_I, \Omega_J, \Omega_K \rangle$. It turns out that **this construction gives a trisymplectic structure, and all trisymplectic structures can be obtained in this way.**

Trisymplectic structures as Mat(2)-representations II

THEOREM: Let V be a vector space equipped with a standard action of the matrix algebra $H \cong$ Mat(2), and $\{I, J, K\}$ a quaternionic basis in Mat(2). Consider the corresponding action of $SL(2)$ on the tensor powers of V. Then, for any $SL(2)$-invariant symmetric form g, denote by $\mathbf{\Omega}$ the space generated by $\Omega_I := g(\cdot, I\cdot)$, Ω_J, Ω_K Then **Ω is a trisymplectic structure on V, with the operators $\Omega_K^{-1} \circ \Omega_J$, $\Omega_K^{-1} \circ \Omega_I$ generating H.** Moreover, **for each trisymplectic structure Ω on V, there exists a unique (up to a constant) $SL(2)$-invariant non-degenerate quadratic form g inducing Ω as above.**

Trisymplectic manifold

DEFINITION: A **trisymplectic structure** on a complex $2n$-manifold M is a triple of holomorphic symplectic forms Ω_1, Ω_2, Ω_3, such that any linear combination of these forms has rank $2n$, n or 0. We denote by $\mathbf{\Omega}$ the 3-dimensional space generated by Ω_i. Obviously, $\mathbf{\Omega}$ defines a trisymplectic structure at each point of M.

REMARK: Let $\Omega_1, \Omega_2 \in \mathbf{\Omega}$. Consider $P(t) := \det(\Omega_1 + t\Omega_2)$ as a polynomial of t. Since the eigenvalues of $\Omega_1 + t\Omega_2$ occur in n-tuples, $P(t) = Q(t)^{n/2}$, **where Q is a quadratic polynomial.**

CLAIM: There exists a non-degenerate quadratic form Q on $\mathbf{\Omega}$, unique up to a constant, such that **$\Omega \in \mathbf{\Omega}$ is degenerate if and only if $Q(\Omega, \Omega) = 0$.**

COROLLARY: For each degenerate $\Omega \in \mathbf{\Omega}$, its radical $\ker \Omega$ is a sub-bundle of codimension n in TM. Moreover, **for all non-proportional degenerate $\Omega, \Omega' \in \mathbf{\Omega}$, one has $TM = \ker \Omega \oplus \ker \Omega'$.**

REMARK: Since Ω is closed, $\ker \Omega$ is **involutive:** $[\ker \Omega, \ker \Omega] \subset \ker \Omega$.

REMARK: Similar to web geometry!

Holomorphic 3-webs.

DEFINITION: Let M be a complex manifold, and S_1, S_2, S_3 integrable, pairwise transversal holomorphic sub-bundles in TM, of dimension $\frac{1}{2}\dim M$. Then (S_1, S_2, S_3) is called **a holomorphic 3-web** on M.

REMARK: On smooth manifolds, the theory of 3-webs is due to Chern and Blaschke (1930-ies).

THEOREM: (Ph. D. thesis of Chern, 1936) Let S_1, S_2, S_3 be a holomorphic 3-web on a complex manifold M. **Then there exists a unique holomorphic connection ∇ on M which preserves the sub-bundles S_i,** and such that its torsion T satisfies $T(S_1, S_2) = 0$.

Holomorphic $SL(2)$-webs.

DEFINITION: A holomorphic 3-web S_1, S_2, S_3 on a complex manifold M is called **an $SL(2)$-web** if

- the projection operators $P_{i,j}$ of TM to S_i along S_j **generate the standard action of** Mat(2) on $\mathbb{C}^2 \otimes \mathbb{C}^n$,
- for any nilpotent $v \in \operatorname{Mat}(2)$, **the bundle $v(TM) \subset TM$ is involutive.**

REMARK: The set of $v \in \operatorname{Mat}(2)$ with $\operatorname{rk} v = 1$ satisfies $\mathbb{P}V = \mathbb{C}P^1$, hence the sub-bundles $v(TM) \subset TM$ are parametrized by $\mathbb{C}P^1$. **An $SL(2)$-web is determined by a set of sub-bundles $S_t \subset TM$, $t \in \mathbb{C}P^1$, which are pairwise transversal and involutive.**

THEOREM: (Jardim–V.) Let $S_t \subset TM$, $t \in \mathbb{C}P^1$ be an $SL(2)$-web on M, and $t_1, t_2, t_3 \in \mathbb{C}P^1$ distinct points. Then the Chern connection of a 3-web S_{t_1}, S_{t_2}, S_{t_3} **is a torsion-free affine holomorphic connection with holonomy in $GL(n,\mathbb{C})$** acting on $\mathbb{C}^{2n} = \mathbb{C}^n \otimes \mathbb{C}^2$, and **independent from the choice of t_i.**

Trisymplectic manifolds

THEOREM: (Jardim–V.) For any trisymplectic structure on M, **the bundles $\ker\Omega \subset TM$ define an $SL(2)$-web.** Moreover, the Chern connection of this $SL(2)$-web **preserves all forms in Ω**.

REMARK: In this case, **the Chern connection has holonomy in $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.**

REMARK: For a trisymplectic structure $\mathbf{\Omega}$, it is just the **Levi-Civita connection of the holomorphic Riemannian form associated with Ω.**

THE REST OF TODAY'S TALK IS EXAMPLES AND APPLICATIONS OF TRISYMPLECTIC GEOMETRY

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: **A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).

Twistor space

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^{=} - \operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Rational curves on Tw(M).

REMARK: The twistor space **has many rational curves**. In fact, it is **rationally connected** (Campana).

DEFINITION: Denote by Sec(M) **the space of holomorphic sections** of the twistor fibration Tw$(M) \xrightarrow{\pi} \mathbb{C}P^1$.

DEFINITION: For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of π. The space of horizontal sections is denoted $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$ is a smooth manifold of dimension $2\dim M$.**

DEFINITION: A twistor section $C \subset \operatorname{Tw}(M)$ is called **regular**, if $NC = \mathcal{O}(1)^{\dim M}$.

CLAIM: For any $I \neq J \in \mathbb{C}P^n$, consider the evaluation map $\operatorname{Sec}(M) \xrightarrow{E_{I,J}} (M, I) \times (M, J)$, $s \longrightarrow s(I) \times s(J)$. Then **$E_{I,J}$ is an isomorphism around the set $\operatorname{Sec}_0(M)$ of regular sections.**

Complexification of a hyperkähler manifold.

REMARK: Consider an anticomplex involution $\operatorname{Tw}(M) \xrightarrow{\iota} \operatorname{Tw}(M)$ mapping (m,t) to $(m,i(t))$, where $i: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\operatorname{Sec}_{hor}(M) = M$ **is a component of the fixed set of** ι**.**

COROLLARY: $\operatorname{Sec}(M)$ **is a complexification of** M**.**

QUESTION: What are geometric structures on $\operatorname{Sec}(M)$?

Answer 1: For compact M, $\operatorname{Sec}(M)$ **is holomorphically convex** (Stein if $\dim M = 2$).

Answer 2: . Let $I \in \mathbb{C}P^1$, and $ev_I: \operatorname{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Then **the 2-forms** $ev_I^* \Omega_I$**,** $I \in \mathbb{C}P^1$ **generate a trisymplectic structure on** $\operatorname{Sec}_0(M)$**.**

Answer 3: The space $\operatorname{Sec}_0(M)$ **admits a holomorphic, torsion-free connection** with holonomy $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

Holomorphic bundles on $\mathbb{C}P^3$ and twistor sections

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

THEOREM: (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

THEOREM: There is a correspondence between the holomorphic bundles on $\operatorname{Tw}(\mathbb{H}) = \mathbb{C}P^3 \backslash \mathbb{C}P^1$, with appropriate stability and framing conditions, and twistor sections in $\operatorname{Sec}(\mathcal{M}_{r,c})$.

It is used to prove the following longstanding conjecture.

THEOREM: (Jardim–V.) **The space $\mathbb{M}_{r,c}$ of framed mathematical instantons on $\mathbb{C}P^3$ is smooth.**

REMARK: To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_{r,c}$ is smooth, we develop **trihyperkähler reduction**, which is **a reduction defined on trisymplectic manifolds.**

Mathematical instantons

DEFINITION: A **mathematical instanton bundle** on $\mathbb{C}P^n$ is a locally free coherent sheaf E on $\mathbb{C}P^n$ with $c_1(E) = 0$ satisfying the following cohomological conditions:
1. for $n \geqslant 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;
2. for $n \geqslant 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;
3. for $n \geqslant 4$, $H^p(E(k)) = 0$, $2 \leqslant p \leqslant n-2$ and $\forall k$;

The integer $c = -\chi(E(-1)) = h^1(E(-1)) = c_2(E)$ is called **the charge** of E. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^n$.

REMARK: Mathematical instantons of rank 2 **are always stable** (follows from the monad description below).

REMARK: The space $\mathbb{M}_{r,c}$ of framed instantons with charge c and rank r **is a principal $SL(2)$-bundle** over the space of all mathematical instantons trivial on ℓ.

THEOREM: (Jardim–V.) The space $\mathbb{M}_c$ of framed rank r mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections** $\operatorname{Sec}(\mathcal{M}_{r,c})$**.**

Monads and mathematical instantons

DEFINITION: A monad is a sequence of vector bundles $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ which is exact in the first and the last term. **The cohomology** of a monad is $\ker j / \operatorname{im} i$.

THEOREM: Let B be a holomorphic bundle of rank 2 on $\mathbb{C}P^n$, $c_1(B) = 0$, $c_2(B) = c$. **Then the following conditions are equivalent.**

(i) B is a mathematical instanton.

(ii) B is a cohomology of a monad

$$0 \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(-1) \longrightarrow W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k} \longrightarrow U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(1) \longrightarrow 0$$

with $\dim V = \dim U = c$ and $\dim W = 2c + 2$.

ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **ADHM data** is maps

$$A, B \in \mathrm{End}(V), I \in \mathrm{Hom}(W, V), J \in \mathrm{Hom}(V, W).$$

We say that ADHM data is

stable,

if there is no subspace $S \subsetneq V$ such that $A(S), B(S) \subset S$ and $I(W) \subset S$;

costable,

if there is no nontrivial subspace $S \subset V$ such that $A(S), B(S) \subset S$ and $S \subset \ker J$;

regular,

if it is both stable and costable.

The ADHM equation is $[A, B] + IJ = 0$.

THEOREM: (Atiyah, Drinfeld, Hitchin, Manin) Framed rank r, charge c instantons on $\mathbb{C}P^2$ are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on $\mathbb{C}P^2$ is identified with moduli of the corresponding quiver representation.**

The multi-dimensional ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **d-dimensional ADHM data** is maps

$$A_k, B_k \in \mathrm{End}(V), I_k \in \mathrm{Hom}(W,V), J_k \in \mathrm{Hom}(V,W), (k = 0, \dots, d)$$

Choose homogeneous coordinates $[z_0 : \dots : z_d]$ on $\mathbb{C}P^d$ and define

$$\tilde{A} := A_0 \otimes z_0 + \dots + A_d \otimes z_d \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + \dots + B_d \otimes z_d.$$

We say that d-dimensional ADHM data is **globally regular**, if $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$ is regular for every $p \in \mathbb{C}P^d$. The **d-dimensional ADHM equation** is $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p\tilde{J}_p = 0$, for all $p \in \mathbb{C}P^d$

THEOREM: (Marcos Jardim, Igor Frenkel) Let $C_d(r,c)$ denote the set of globally regular solutions of the d-dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the d-dimensional ADHM equations and isomorphism classes of rank r instanton bundles** on $\mathbb{CP}^{d+2}$ framed at a fixed line ℓ, where $\dim W = \mathrm{rk}(E)$ and $\dim V = c_2(E)$.

The multi-dimensional ADHM construction for $d=1$

For $d=1$, we obtain that the d-dimensional ADHM solutions are families of solutions of ADHM parametrized by $\mathbb{C}P^3$. Also, the space of 1-dimensional ADHM data is the space of sections of

$$\mathcal{O}(1)\otimes_{\mathbb{C}}\left[\mathrm{Hom}(W,V)\oplus\mathrm{Hom}(V,W)\oplus\mathrm{End}(V)\oplus\mathrm{End}(V)\right]$$

over $\mathbb{C}P^1$, that is, the twistor space of a qquaternionic vector space $U=\mathrm{Hom}(W,V)\oplus\mathrm{Hom}(V,W)\oplus\mathrm{End}(V)\oplus\mathrm{End}(V)$. Now, the hyperkähler structure on 0-dimensional ADHM solutions for each $p\in\mathbb{C}P^1$ is compatible with the hyperkähler structure on U, because the space of 0-dimensional ADHM solutions is obtained from U by hyperkähler reduction. **This is used to prove the theorem about instantons on $\mathbb{C}P^3$ and twistor sections.**