# Trisymplectic manifolds

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## Complexification of a manifold

**DEFINITION:** Let $M$ be a complex manifold, equipped with an anticomplex involution $\iota$. The fixed point set $M_{\mathbb{R}}$ of $\iota$ is called **a real analytic manifold**, and a germ of $M$ in $M_{\mathbb{R}}$ is called **a complexification** of $M_{\mathbb{R}}$.

**QUESTION: What is a complexification of a Kähler manifold** (considered as real analytic variety)?

**THEOREM:** (D. Kaledin, B. Feix) Let $M$ be a real analytic Kähler manifold, and $M_{\mathbb{C}}$ its complexification. **Then $M_{\mathbb{C}}$ admits a hyperkähler structure,** determined uniquely and functorially by the Kähler structure on $M$.

**QUESTION: What is a complexification of a hyperkähler manifold?**

THIS IS THE MAIN SUBJECT OF TODAY'S TALK.

**(A joint work with Marcos Jardim).**

**Plan of the talk:**

1. Trisymplectic structures on a vector space (linear algebra).

2. Trisymplectic structures on a manifold (differential geometry).

3. Trisymplectic structure on the space of rational lines in the twistor space (hyperkähler geometry).

4. Applications to the instanton spaces.

## Trisymplectic structure on a vector space

**DEFINITION:** A **trisymplectic structure** on a complex vector space of dimension $2n$ is a 3-dimensional space $\mathbf{\Omega} \subset \Lambda^2 V$ of complex linear 2-forms, such that any $\eta \in \mathbf{\Omega}$ has rank $2n$, $n$ or 0.

**REMARK:** It is easy to see that **$\Omega$ contains a symplectic form.**

**PROPOSITION:** Given two symplectic forms $\omega_1, \omega_2 \in \mathbf{\Omega}$, consider the map $\varphi_{\Omega_1,\Omega_2} := \omega_1 \circ \omega_2^{-1} \in \operatorname{End}(V)$. Then $\varphi_{\Omega_1,\Omega_2}$ can be expressed in an appropriate basis by the matrix

$$\varphi_{\omega_1,\omega_2} = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda' & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda' \end{pmatrix},$$

with the eigenspaces of equal dimension.

**THEOREM:** Let $(V, \mathbf{\Omega})$ be a be a trisymplectic vector space, and $H \subset \operatorname{End}(V)$ the algebra generated by $\varphi_{\Omega_1,\Omega_2}$, for all $\omega_1, \omega_2 \in \mathbf{\Omega}$. **Then $H$ is isomorphic to the matrix algebra $\operatorname{Mat}(2)$,** acting on $V$ in a standard way.

## Trisymplectic structures as Mat(2)-representations

**DEFINITION:** Let $V$ be a complex vector space with the **standard action** of the matrix algebra Mat(2), i.e. $V \cong V_0 \otimes \mathbb{C}^2$ and Mat(2) acts only through the second factor.

**CLAIM:** Consider the natural $SL(2)$-action on $V$ induced by Mat(2), and extend it multiplicatively to all tensor powers of $V$. Let $g \in \operatorname{Sym}^2_{\mathbb{C}}(V)$ be an $SL(2)$-invariant, non-degenerate 2-form on $V$, and $\{I, J, K\}$ a quaternionic basis in Mat(2) Then

$$g(x, Iy) = g(Ix, I^2 y) = -g(Ix, y)$$

hence the form $\Omega_I(\cdot,\cdot) := g(\cdot, I\cdot)$ is a symplectic form, obviously non-degenerate; the forms $\Omega_J$, $\Omega_K$ have the same properties. Let $\mathbf{\Omega} := \langle \Omega_I, \Omega_J, \Omega_K \rangle$. It turns out that **this construction gives a trisymplectic structure, and all trisymplectic structures can be obtained in this way.**

## Trisymplectic structures as Mat(2)-representations II

**THEOREM:** Let $V$ be a vector space equipped with a standard action of the matrix algebra $H \cong$ Mat(2), and $\{I, J, K\}$ a quaternionic basis in Mat(2). Consider the corresponding action of $SL(2)$ on the tensor powers of $V$. Then, for any $SL(2)$-invariant symmetric form $g$, denote by $\mathbf{\Omega}$ the space generated by $\Omega_I := g(\cdot, I\cdot)$, $\Omega_J$, $\Omega_K$ Then **$\Omega$ is a trisymplectic structure on $V$, with the operators $\Omega_K^{-1} \circ \Omega_J$, $\Omega_K^{-1} \circ \Omega_I$ generating $H$.** Moreover, **for each trisymplectic structure $\Omega$ on $V$, there exists a unique (up to a constant) $SL(2)$-invariant non-degenerate quadratic form $g$ inducing $\Omega$ as above.**

## Trisymplectic manifold

**DEFINITION:** A **trisymplectic structure** on a complex $2n$-manifold $M$ is a triple of holomorphic symplectic forms $\Omega_1$, $\Omega_2$, $\Omega_3$, such that any linear combination of these forms has rank $2n$, $n$ or 0. We denote by $\mathbf{\Omega}$ the 3-dimensional space generated by $\Omega_i$. Obviously, $\mathbf{\Omega}$ defines a trisymplectic structure at each point of $M$.

**REMARK:** Let $\Omega_1, \Omega_2 \in \mathbf{\Omega}$. Consider $P(t) := \det(\Omega_1 + t\Omega_2)$ as a polynomial of $t$. Since the eigenvalues of $\Omega_1 + t\Omega_2$ occur in $n$-tuples, **$P(t) = Q(t)^{n/2}$, where $Q$ is a quadratic polynomial.**

**CLAIM:** There exists a non-degenerate quadratic form $Q$ on $\mathbf{\Omega}$, unique up to a constant, such that **$\Omega \in \mathbf{\Omega}$ is degenerate if and only if $Q(\Omega, \Omega) = 0$.**

**COROLLARY:** For each degenerate $\Omega \in \mathbf{\Omega}$, its radical $\ker \Omega$ is a sub-bundle of codimension $n$ in $TM$. Moreover, **for all non-proportional degenerate $\Omega, \Omega' \in \mathbf{\Omega}$, one has $TM = \ker \Omega \oplus \ker \Omega'$.**

**REMARK:** Since $\Omega$ is closed, $\ker \Omega$ is **involutive:** $[\ker \Omega, \ker \Omega] \subset \ker \Omega$.

**REMARK:** Similar to web geometry!

## Holomorphic 3-webs.

**DEFINITION:** Let $M$ be a complex manifold, and $S_1$, $S_2$, $S_3$ integrable, pairwise transversal holomorphic sub-bundles in $TM$, of dimension $\frac{1}{2}\dim M$. Then $(S_1, S_2, S_3)$ is called **a holomorphic 3-web** on $M$.

**REMARK:** On smooth manifolds, the theory of 3-webs is due to Chern and Blaschke (1930-ies).

**THEOREM:** (Ph. D. thesis of Chern, 1936) Let $S_1, S_2, S_3$ be a holomorphic 3-web on a complex manifold $M$. **Then there exists a unique holomorphic connection $\nabla$ on $M$ which preserves the sub-bundles $S_i$,** and such that its torsion $T$ satisfies $T(S_1, S_2) = 0$.

## Holomorphic $SL(2)$-webs.

**DEFINITION:** A holomorphic 3-web $S_1$, $S_2$, $S_3$ on a complex manifold $M$ is called **an $SL(2)$-web** if

- the projection operators $P_{i,j}$ of $TM$ to $S_i$ along $S_j$ **generate the standard action of** $\mathrm{Mat}(2)$ on $\mathbb{C}^2 \otimes \mathbb{C}^n$,
- for any nilpotent $v \in \mathrm{Mat}(2)$, **the bundle $v(TM) \subset TM$ is involutive.**

**REMARK:** The set of $v \in \mathrm{Mat}(2)$ with $\operatorname{rk} v = 1$ satisfies $\mathbb{P}V = \mathbb{C}P^1$, hence the sub-bundles $v(TM) \subset TM$ are parametrized by $\mathbb{C}P^1$. **An $SL(2)$-web is determined by a set of sub-bundles $S_t \subset TM$, $t \in \mathbb{C}P^1$, which are pairwise transversal and involutive.**

**THEOREM:** (Jardim–V.) Let $S_t \subset TM$, $t \in \mathbb{C}P^1$ be an $SL(2)$-web on $M$, and $t_1, t_2, t_3 \in \mathbb{C}P^1$ distinct points. Then the Chern connection of a 3-web $S_{t_1}$, $S_{t_2}$, $S_{t_3}$ **is a torsion-free affine holomorphic connection with holonomy in $GL(n,\mathbb{C})$** acting on $\mathbb{C}^{2n} = \mathbb{C}^n \otimes \mathbb{C}^2$, and **independent from the choice of $t_i$.**

## Trisymplectic manifolds

**THEOREM:** (Jardim–V.) For any trisymplectic structure on $M$, **the bundles $\ker\Omega \subset TM$ define an $SL(2)$-web.** Moreover, the Chern connection of this $SL(2)$-web **preserves all forms in $\Omega$.**

**REMARK:** In this case, **the Chern connection has holonomy in $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.**

**REMARK:** For a trisymplectic structure $\Omega$, it is just the **Levi-Civita connection of the holomorphic Riemannian form associated with $\Omega$.**

**THE REST OF TODAY'S TALK IS EXAMPLES AND APPLICATIONS OF TRISYMPLECTIC GEOMETRY**

## Hyperkähler manifolds

**DEFINITION:** A **hyperkähler structure** on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of $M$.

**REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving $I, J, K$).

## Twistor space

**DEFINITION: Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if $M$ is compact, for generic $a, b, c$, $(M, L)$ has no divisors (Fujiki).

**DEFINITION:** A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

**EXAMPLE:** If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

**REMARK: For $M$ compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.**

## Rational curves on $\operatorname{Tw}(M)$.

**REMARK:** The twistor space **has many rational curves**. In fact, it is **rationally connected** (Campana).

**DEFINITION:** Denote by $\operatorname{Sec}(M)$ **the space of holomorphic sections** of the twistor fibration $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$.

**DEFINITION:** For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of $\pi$. The space of horizontal sections is denoted $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$

**REMARK:** The space of horizontal sections of $\pi$ is identified with $M$. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$ is a smooth manifold of dimension $2\dim M$.**

**DEFINITION:** A twistor section $C \subset \operatorname{Tw}(M)$ is called **regular**, if $NC = \mathcal{O}(1)^{\dim M}$.

**CLAIM:** For any $I \neq J \in \mathbb{C}P^n$, consider the evaluation map $\operatorname{Sec}(M) \xrightarrow{E_{I,J}} (M, I) \times (M, J)$, $s \longrightarrow s(I) \times s(J)$. Then **$E_{I,J}$ is an isomorphism around the set $\operatorname{Sec}_0(M)$ of regular sections.**

## Complexification of a hyperkähler manifold.

**REMARK:** Consider an anticomplex involution $\operatorname{Tw}(M) \xrightarrow{\iota} \operatorname{Tw}(M)$ mapping $(m,t)$ to $(m,i(t))$, where $i: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\operatorname{Sec}_{hor}(M) = M$ **is a component of the fixed set of $\iota$.**

**COROLLARY:** $\operatorname{Sec}(M)$ **is a complexification of $M$.**

**QUESTION:** What are geometric structures on $\operatorname{Sec}(M)$?

**Answer 1:** For compact $M$, $\operatorname{Sec}(M)$ **is holomorphically convex** (Stein if $\dim M = 2$).

**Answer 2:** . Let $I \in \mathbb{C}P^1$, and $ev_I: \operatorname{Sec}_0(M) \longrightarrow (M,I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Then **the 2-forms $ev_I^*\Omega_I$, $I \in \mathbb{C}P^1$ generate a trisymplectic structure on $\operatorname{Sec}_0(M)$.**

**Answer 3:** The space $\operatorname{Sec}_0(M)$ **admits a holomorphic, torsion-free connection** with holonomy $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

## Holomorphic bundles on $\mathbb{C}P^3$ and twistor sections

**DEFINITION: An instanton** on $\mathbb{C}P^2$ is a stable bundle $B$ with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

**THEOREM:** (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

**THEOREM: There is a correspondence** between the holomorphic bundles on $\operatorname{Tw}(\mathbb{H}) = \mathbb{C}P^3 \backslash \mathbb{C}P^1$, with appropriate stability and framing conditions, and twistor sections in $\operatorname{Sec}(\mathcal{M}_{r,c})$.

It is used to prove the following longstanding conjecture.

**THEOREM:** (Jardim–V.) **The space $\mathbb{M}_{r,c}$ of framed mathematical instantons on $\mathbb{C}P^3$ is smooth.**

**REMARK:** To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_{r,c}$ is smooth, we develop **trihyperkähler reduction**, which is **a reduction defined on trisymplectic manifolds.**

## Mathematical instantons

**DEFINITION:** A **mathematical instanton bundle** on $\mathbb{C}P^n$ is a locally free coherent sheaf $E$ on $\mathbb{C}P^n$ with $c_1(E) = 0$ satisfying the following cohomological conditions:
1. for $n \geqslant 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;
2. for $n \geqslant 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;
3. for $n \geqslant 4$, $H^p(E(k)) = 0$, $2 \leqslant p \leqslant n-2$ and $\forall k$;

The integer $c = -\chi(E(-1)) = h^1(E(-1)) = c_2(E)$ is called **the charge** of $E$. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^n$.

**REMARK:** Mathematical instantons of rank 2 **are always stable** (follows from the monad description below).

**REMARK:** The space $\mathbb{M}_{r,c}$ of framed instantons with charge $c$ and rank $r$ **is a principal $SL(2)$-bundle** over the space of all mathematical instantons trivial on $\ell$.

**THEOREM:** (Jardim–V.) The space $\mathbb{M}_c$ of framed rank $r$ mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections** $\mathrm{Sec}(\mathcal{M}_{r,c})$.

## Monads and mathematical instantons

**DEFINITION: A monad** is a sequence of vector bundles $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ which is exact in the first and the last term. **The cohomology** of a monad is $\ker j / \operatorname{im} i$.

**THEOREM:** Let $B$ be a holomorphic bundle of rank 2 on $\mathbb{C}P^n$, $c_1(B) = 0$, $c_2(B) = c$. **Then the following conditions are equivalent.**

(i) $B$ is a mathematical instanton.

(ii) $B$ is a cohomology of a monad

$$0 \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(-1) \longrightarrow W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k} \longrightarrow U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(1) \longrightarrow 0$$

with $\dim V = \dim U = c$ and $\dim W = 2c + 2$.

## ADHM construction

**DEFINITION:** Let $V$ and $W$ be complex vector spaces, with dimensions $c$ and $r$, respectively. The **ADHM data** is maps

$$A, B \in \mathrm{End}(V), I \in \mathrm{Hom}(W, V), J \in \mathrm{Hom}(V, W).$$

We say that ADHM data is

**stable**,

if there is no subspace $S \subsetneq V$ such that $A(S), B(S) \subset S$ and $I(W) \subset S$;

**costable**,

if there is no nontrivial subspace $S \subset V$ such that $A(S), B(S) \subset S$ and $S \subset \ker J$;

**regular**,

if it is both stable and costable.

**The ADHM equation** is $[A, B] + IJ = 0$.

**THEOREM:** (Atiyah, Drinfeld, Hitchin, Manin) Framed rank $r$, charge $c$ instantons on $\mathbb{C}P^2$ are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on $\mathbb{C}P^2$ is identified with moduli of the corresponding quiver representation.**

## The multi-dimensional ADHM construction

**DEFINITION:** Let $V$ and $W$ be complex vector spaces, with dimensions $c$ and $r$, respectively. The **$d$-dimensional ADHM data** is maps

$$A_k, B_k \in \operatorname{End}(V), I_k \in \operatorname{Hom}(W,V), J_k \in \operatorname{Hom}(V,W), (k = 0, \dots, d)$$

Choose homogeneous coordinates $[z_0 : \dots : z_d]$ on $\mathbb{C}P^d$ and define

$$\tilde{A} := A_0 \otimes z_0 + \dots + A_d \otimes z_d \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + \dots + B_d \otimes z_d.$$

We say that $d$-dimensional ADHM data is **globally regular**, if $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$ is regular for every $p \in \mathbb{C}P^d$. The **$d$-dimensional ADHM equation** is $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p = 0$, for all $p \in \mathbb{C}P^d$

**THEOREM:** (Marcos Jardim, Igor Frenkel) Let $C_d(r,c)$ denote the set of globally regular solutions of the $d$-dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the $d$-dimensional ADHM equations and isomorphism classes of rank $r$ instanton bundles** on $\mathbb{CP}^{d+2}$ framed at a fixed line $\ell$, where $\dim W = \operatorname{rk}(E)$ and $\dim V = c_2(E)$.

## The multi-dimensional ADHM construction for $d = 1$

For $d = 1$, we obtain that the $d$-dimensional ADHM solutions are families of solutions of ADHM parametrized by $\mathbb{C}P^3$. Also, the space of 1-dimensional ADHM data is the space of sections of

$$\mathcal{O}(1) \otimes_{\mathbb{C}} \left[ \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V) \right]$$

over $\mathbb{C}P^1$, that is, the twistor space of a qquaternionic vector space $U = \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V)$. Now, the hyperkähler structure on 0-dimensional ADHM solutions for each $p \in \mathbb{C}P^1$ is compatible with the hyperkähler structure on $U$, because the space of 0-dimensional ADHM solutions is obtained from $U$ by hyperkähler reduction. **This is used to prove the theorem about instantons on $\mathbb{C}P^3$ and twistor sections.**