Stable bundles on $\mathbb{C}P^3$ and special holonomies

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Géométrie des variétés complexes IV

CIRM, Luminy, Oct 26, 2010

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

EXAMPLE: An even-dimensional complex torus.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

DEFINITION: A **K3 surface** is a complex 2-manifold obtained as a deformation of a Kummer surface.

REMARK: **A K3 surface is always hyperkähler.** Any hyperkähler manifold of real dimension 4 is isomorphic to a torus or a K3 surface.

Complexification of a manifold

DEFINITION: Let M be a complex manifold, equipped with an anticomplex involution ι. The fixed point set $M_{\mathbb{R}}$ of ι is called **a real analytic manifold**, and a germ of M in $M_{\mathbb{R}}$ is called **a complexification** of $M_{\mathbb{R}}$.

QUESTION: What is a complexification of a Kähler manifold (considered as real analytic variety)?

THEOREM: (D. Kaledin, B. Feix) Let M be a real analytic Kähler manifold, and $M_{\mathbb{C}}$ its complexification. **Then $M_{\mathbb{C}}$ admits a hyperkähler structure,** determined uniquely and functorially by the Kähler structure on M.

QUESTION: What is a complexification of a hyperkähler manifold?

THIS IS THE MAIN SUBJECT OF TODAY'S TALK.

(A joint work with Marcos Jardim).

Twistor space

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\text{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \to T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. **It defines an almost complex structure on $\text{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\text{Tw}(M) = \text{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\text{Tw}(M)$ never admits a Kähler structure.

Rational curves on Tw(M).

REMARK: The twistor space **has many rational curves**. In fact, it is **rationally connected** (Campana).

DEFINITION: Denote by Sec(M) **the space of holomorphic sections** of the twistor fibration Tw$(M) \xrightarrow{\pi} \mathbb{C}P^1$.

DEFINITION: For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of π. The space of horizontal sections is denoted Sec$_{hor}(M) \subset$ Sec(M)

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of Sec$_{hor}(M) \subset$ Sec(M) is a smooth manifold of dimension $2\dim M$.**

DEFINITION: A twistor section $C \subset$ Tw(M) is called **regular**, if $NC = \mathcal{O}(1)^{\dim M}$.

CLAIM: For any $I \neq J \in \mathbb{C}P^n$, consider the evaluation map Sec$(M) \xrightarrow{E_{I,J}} (M,I) \times (M,J)$, $s \longrightarrow s(I) \times s(J)$. Then **$E_{I,J}$ is an isomorphism around the set Sec$_0(M)$ of regular sections.**

Complexification of a hyperkähler manifold.

REMARK: Consider an anticomplex involution $\operatorname{Tw}(M) \xrightarrow{\iota} \operatorname{Tw}(M)$ mapping (m,t) to $(m,i(t))$, where $i: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\operatorname{Sec}_{hor}(M) = M$ **is a component of the fixed set of** ι**.**

COROLLARY: $\operatorname{Sec}(M)$ **is a complexification of** M**.**

QUESTION: What are geometric structures on $\operatorname{Sec}(M)$?

Answer 1: For compact M, $\operatorname{Sec}(M)$ **is holomorphically convex** (Stein if $\dim M = 2$).

Answer 2: The space $\operatorname{Sec}_0(M)$ **admits a holomorphic, torsion-free connection** with holonomy $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$. **This is the "special holonomy" mentioned in the title of the talk.**

REMARK: Merkulov, Schwachhöfer: classification of irreducible special holonomy. $Sp(n,\mathbb{C})$-action on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ is **non-irreducible**.

Mathematical instantons

DEFINITION: A mathematical instanton on $\mathbb{C}P^3$ is a stable rank 2 bundle B with $c_1(B) = 0$ and $H^1(B(-1)) = 0$. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^3$.

REMARK: The space $\mathbb{M}_c$ of framed instantons with $c_2 = c$ **is a principal $SL(2)$-bundle** over the space of all mathematical instantons trivial on ℓ.

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_x$ for some fixed point $x \in \mathbb{C}P^2$.

THEOREM: (Atiyah-Drinfeld-Hitchin-Manin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

THEOREM: (Jardim–V.) The space $\mathbb{M}_c$ of framed mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections $\mathrm{Sec}(\mathcal{M}_{2,c})$.**

The space of instantons on $\mathbb{C}P^3$

CONJECTURE: **The space of mathematical instantons is smooth and connected.**

THEOREM: (Grauert-Müllich, Hauzer-Langer) **Every mathematical instanton on $\mathbb{C}P^3$ is trivial on some line $\ell \subset \mathbb{C}P^3$.**

COROLLARY: The space of mathematical instantons **is covered by Zariski open, dense subvarieties** which take form $\mathbb{M}_c/SL(2,\mathbb{C})$.

COROLLARY: To prove that the space of mathematical instantons is smooth and connected **it would suffice to prove it for $\mathbb{M}_c$.**

THEOREM: (Jardim–V.) **The space $\mathbb{M}_c$ is smooth and connected.**

REMARK: To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_c$ is smooth and connected, we develop **trihyperkähler reduction**, which is **a reduction defined on manifolds with holonomy in $Sp(n,\mathbb{C})$ acting $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.**

We prove that $\mathbb{M}_c$ **is a trihyperkähler quotient** of a vector space by a reductive group action, hence smooth.

Holomorphic 3-webs.

DEFINITION: Let M be a complex manifold, and S_1, S_2, S_3 integrable, pairwise transversal holomorphic sub-bundles in TM, of dimension $\frac{1}{2}\dim M$. Then (S_1, S_2, S_3) is called **a holomorphic 3-web** on M.

REMARK: On smooth manifolds, the theory of 3-webs is due to Chern and Blaschke (1930-ies).

THEOREM: (Ph. D. thesis of Chern, 1936) Let S_1, S_2, S_3 be a holomorphic 3-web on a complex manifold M. **Then there exists a unique holomorphic connection ∇ on M which preserves the sub-bundles S_i,** and such that its torsion T satisfies $T(S_1, S_2) = 0$.

Holomorphic $SL(2)$-webs.

DEFINITION: A holomorphic 3-web on a complex manifold M is called **an $SL(2)$-web** if
- the projection operators $P_{i,j}$ of TM to S_i along S_j **generate the standard action of** Mat(2) on $\mathbb{C}^2 \otimes \mathbb{C}^n$,
- for any nilpotent $v \in$ Mat(2), **the bundle $v(TM) \subset TM$ is involutive.**

REMARK: The set of $v \in$ Mat(2) with $\operatorname{rk} v = 1$ satisfies $\mathbb{P}V = \mathbb{C}P^1$, hence the sub-bundles $v(TM) \subset TM$ are parametrized by $\mathbb{C}P^1$. **An $SL(2)$-web is determined by a set of sub-bundles $S_t \subset TM$, $t \in \mathbb{C}P^1$, which are pairwise transversal and involutive.**

EXAMPLE: Consider a hyperkähler manifold M. Let $I \in \mathbb{C}P^1$, and ev_I : $\operatorname{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Then $\ker Dev_I$, $I \in \mathbb{C}P^1$ **is an $SL(2)$-web.**

THEOREM: (Jardim–V.) Let $S_t \subset TM$, $t \in \mathbb{C}P^1$ be an $SL(2)$-web on M, and $t_1, t_2, t_3 \in \mathbb{C}P^1$ distinct points. Then the Chern connection of a 3-web S_{t_1}, S_{t_2}, S_{t_3} **is a torsion-free affine holomorphic connection with holonomy in $GL(n, \mathbb{C})$** acting on $\mathbb{C}^{2n} = \mathbb{C}^n \otimes \mathbb{C}^2$, and **independent from the choice of t_i.**

Trisymplectic manifolds

DEFINITION: Let $\mathbf{\Omega}$ be a 3-dimensional space of holomorphic symplectic 2-forms on a manifold. Suppose that

- **$\mathbf{\Omega}$ contains a non-degenerate 2-form**
- For each non-zero degenerate $\Omega \in \mathbf{\Omega}$, one has $\operatorname{rk}\Omega = \frac{1}{2}\dim V$.

Then $\mathbf{\Omega}$ is called **a trisymplectic structure on M**.

REMARK: The bundles $\ker\Omega$ are involutive, because Ω is closed.

REMARK: This notion is similar to **hypersymplectic structures** (which are a triple of closed forms on a real manifold with the same rank condition).

THEOREM: (Jardim–V.) For any trisymplectic structure on M, **the bundles $\ker\Omega \subset TM$ define an $SL(2)$-web.** Moreover, the Chern connection of this $SL(2)$-web **preserves all forms in $\mathbf{\Omega}$**.

REMARK: In this case, **the Chern connection has holonomy in $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.**

Trisymplectic structure on $\operatorname{Sec}_0(M)$

EXAMPLE: Consider a hyperkähler manifold M. Let $I \in \mathbb{C}P^1$, and $ev_I : \operatorname{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Denote by Ω_I the holomorphic symplectic form on (M, I). **Then $ev_I^* \Omega_I$, $I \in \mathbb{C}P^1$ generate a trisymplectic structure.**

COROLLARY: $\operatorname{Sec}_0(M)$ **is equipped with a holomorphic, torsion-free connection with holonomy in $Sp(n, \mathbb{C})$.**

Hyperkähler reduction

DEFINITION: Let G be a compact Lie group acting on a hyperkähler manifold M by hyperkähler isometries. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g} \otimes \mathbb{R}^3$ such that $\langle d\mu_i(v), \xi \rangle = \omega_i(\xi^*, v)$, for every $v \in TM$, $\xi \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one the the Kähler forms associated with the hyperkähler structure.

DEFINITION: The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)**The quotient $M /\!\!/\!\!/ G$ is hyperkaehler**.

Trihyperkähler reduction

DEFINITION: A trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*$ takes vectors $\Omega \in \boldsymbol{\Omega}, g \in \mathfrak{g} = \operatorname{Lie}(G)$ and maps them to a holomorphic function $f \in \mathcal{O}_M$, such that $df = \Omega \lrcorner g$, where $\Omega \lrcorner g$ denotes the contraction of Ω and the vector field g

DEFINITION: Let $(M, \boldsymbol{\Omega}, S_t)$ be a trisymplectic structure on a complex manifold M. Assume that M is equipped with an action of a compact Lie group G preserving $\boldsymbol{\Omega}$, and an equivariant trisymplectic moment map

$$\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*.$$

Let $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ be the corresponding **level set** of the moment map. Consider the action of the complex Lie group $G_{\mathbb{C}}$ on $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$. Assume that it is proper and free. Then the quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$ is a smooth manifold called **the trisymplectic quotient** of $(M, \boldsymbol{\Omega}, S_t)$, denoted by $M /\!\!/\!\!/\!\!/ G$.

THEOREM: Suppose that the restriction of $\boldsymbol{\Omega}$ to $\mathfrak{g} \subset TM$ is non-degenerate. **Then $M /\!\!/\!\!/\!\!/ G$ trisymplectic.**

Hyperholomorphic connections

REMARK: Let M be a hyperkähler manifold. **The group $SU(2)$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda_I^{1,1}(M)$, **a hyperholomorphic connection defines a holomorphic structure on B** for each I induced by quaternions.

REMARK: Let M be a compact hyperkähler manifold. Then $SU(2)$ preserves harmonic forms, hence **acts on cohomology.**

Hyperholomorphic bundles and twistor sections

THEOREM: (V., 1995) Let B be a stable bundle on a compact hyperkähler manifold with $c_1(B)$ and $c_2(B)$ $SU(2)$-invariant. **Then B admits a unique hyperholomorphic connection.**

DEFINITION: A stable bundle with $c_1(B)$ and $c_2(B)$ $SU(2)$-invariant is called **hyperholomorphic**.

COROLLARY: The space of deformations of a hyperholomorphic bundle is a hyperkähler manifold.

COROLLARY: A hyperholomorphic bundle can be lifted to a holomorphic bundle $\mathcal{B}$ on a twistor space.

THEOREM: (Kaledin–V., 1996) **The space** $\mathrm{Sec}_0(\mathrm{Def}(B))$ **admits an open embedding to a space** $\mathrm{Def}(\mathcal{B})$ of deformations of $\mathcal{B}$ on $\mathrm{Tw}(M)$, and its image is Zariski dense.

REMARK: Let $\mathcal{M}_{2,c}$ be the space of framed instantons on $\mathbb{C}^2$. The above theorem gives **an embedding from** $\mathrm{Sec}_0(\mathcal{M}_{2,c})$ **to the space of holomorphic bundles on** $\mathrm{Tw}(\mathbb{C}^2) = \mathbb{C}P^3 \backslash \mathbb{C}P^1$.

Hyperholomorphic bundles and mathematical instantons

REMARK: Using the monad description of mathematical instantons, **we prove that that the map $\operatorname{Sec}_0(\mathcal{M}_{2,c}) \longrightarrow \mathbb{M}_c$ to the space of mathematical instantons is an isomorphism** (Frenkel-Jardim, Jardim-V.).

REMARK: The smoothness of the space $\operatorname{Sec}_0(\mathcal{M}_{2,c}) = \mathbb{M}_c$ **follows from the trihyperkähler reduction procedure:**

THEOREM: Let M be flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient $M /\!\!/\!\!/ G$ is smooth. **Then there exists an open embedding**

$$\operatorname{Sec}_0(M) /\!\!/\!\!/\!\!/ G \xrightarrow{\Psi} \operatorname{Sec}_0(M /\!\!/\!\!/ G),$$

which is compatible with the trisymplectic structures on $\operatorname{Sec}_0(M) /\!\!/\!\!/\!\!/ G$ and $\operatorname{Sec}_0(M /\!\!/\!\!/ G)$.

THEOREM: If M is the quiver space which gives $M /\!\!/\!\!/ G = \mathcal{M}_{2,c}$, **$\Psi$ gives an isomorphism $\operatorname{Sec}_0(M) /\!\!/\!\!/\!\!/ G = \operatorname{Sec}_0(M /\!\!/\!\!/ G)$.**