# Trisymplectic manifolds and the moduli of instantons on $\mathbb{C}P^3$

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## Plan

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## Moment maps

**DEFINITION:** $(M,\omega)$ be a symplectic manifold, $G$ a Lie group acting on $M$ by symplectomorphisms. **A moment map** $\mu$ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

**REMARK:** It is more convenient to consider $\mu$ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

**REMARK:** Moment map **always exists** if $M$ is simply connected.

**DEFINITION:** A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of $G$ on $\mathfrak{g}^*$.

**REMARK:** $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is $G$-invariant**.

**DEFINITION:** A $G$-invariant $c \in \mathfrak{g}^*$ is called **central**.

**CLAIM: An equivariant moment map exists whenever $H^1(G, \mathfrak{g}^*) = 0$.** In particular, when $G$ is reductive and $M$ is simply connected, an equivariant moment map exists.

## Symplectic reduction and GIT

**DEFINITION:** (Weinstein-Marsden) $(M, \omega)$ be a symplectic manifold, $G$ a compact Lie group acting on $M$ by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of $M$, denoted by $M/\!\!/G$.

**CLAIM:** The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

**THEOREM:** Let $(M, I, \omega)$ be a Kähler manifold, $G_{\mathbb{C}}$ a complex reductive Lie group acting on $M$ by holomorphic automorphisms, and $G$ is compact form acting isometrically. **Then $M/\!\!/G$ is a Kähler orbifold.**

**REMARK:** In such a situation, $M/\!\!/G$ is called **the Kähler quotient**, or **GIT quotient**. The choice of a central element $c \in \mathfrak{g}^*$ is known as a choice of **stability data**.

**REMARK:** **The points of $M/\!\!/G$ are in bijective correspondence with the orbits of $G_{\mathbb{C}}$ which intersect $\mu^{-1}(c)$.** Such orbits are called **polystable**, and the intersection of a $G_{\mathbb{C}}$-orbit with $\mu^{-1}(c)$ is a $G$-orbit.

## Hyperkähler manifolds

**DEFINITION:** A **hyperkähler structure** on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**REMARK:**
The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is **holomorphic and symplectic** on $(M, I)$.

## Hyperkähler reduction

**DEFINITION:** Let $G$ be a compact Lie group, $\rho$ its action on a hyperkähler manifold $M$ by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a $G$-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g\rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where $\omega_i$ is one three Kähler forms associated with the hyperkähler structure.

**DEFINITION:** Let $\xi_1, \xi_2, \xi_3$ be three $G$-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of $M$.

**THEOREM:** (Hitchin, Karlhede, Lindström, Roček)
**The quotient $M /\!\!/\!\!/ G$ is hyperkaehler**.

## Quiver representations

**DEFINITION:** A **quiver** is an oriented graph. A **quiver representation** is a diagram of complex Hermitian vector spaces and arrows associated with a quiver:

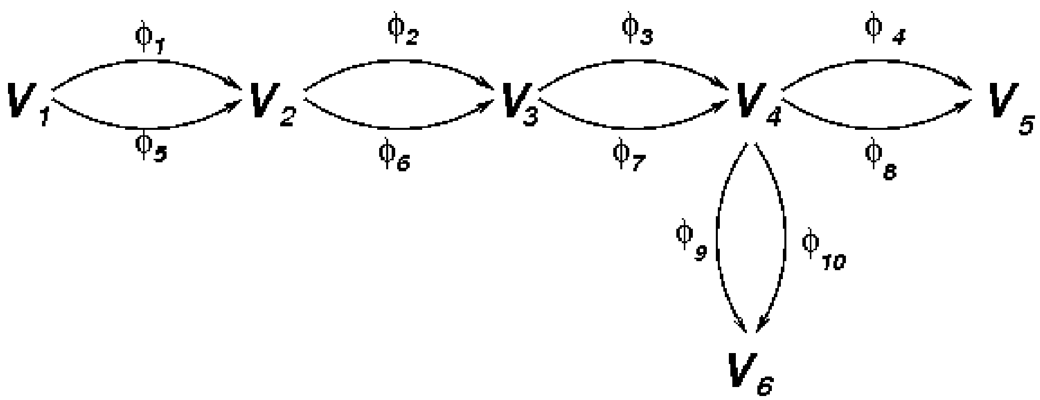
$$\begin{tikzcd} V_1 \arrow[r, "\phi_1"] & V_2 \arrow[r, "\phi_2"] & V_3 \arrow[r, "\phi_3"] & V_4 \arrow[r, "\phi_4"] \arrow[d, "\phi_9"'] & V_5 \\ & & & V_6 & \end{tikzcd}$$

Here, $V_i$ are vector spaces, and $\varphi_i$ linear maps.

**REMARK:** If one fixes the spaces $V_i$, the space of quiver representations is a Hermitian vector space.

## Quiver varieties

Starting from a single graph, one can double it up, as follows, obtaining a Nakajima double quiver.




A Nakajima quiver for the Dynkin diagram $D_5$.

**CLAIM:** The space $M$ of representations of a Nakajima's double quiver is a quaternionic vector space, and the group $G := U(V_1) \times U(V_2) \times ... \times U(V_n)$ acts on $M$ preserving the quaternionic structure.

**DEFINITION:** A **Nakajima quiver variety** is a quotient $M /\!/\!/ G$.

## Hyperkähler manifolds as quiver varieties

Many non-compact hyperkähler manifolds are obtained as quiver varieties.

**EXAMPLE:** A 4-dimensional ALE **(asymptotically locally Euclidean)** space obtained as a resolution of **a du Val singularity**, that is, a quotient $\mathbb{C}^2/G$, where $G \subset SU(2)$ is a finite group.

**REMARK:** Since finite subgroups of $SU(2)$ are classified by the Dynkin diagrams of type A,D,E, these ALE quotients are called **ALE spaces of A-D-E type**.

**EXAMPLE:** The moduli asymptotically flat Hermitian Yang-Mills connections on ALE spaces of A-D-E type.

**DEFINITION: An instanton** on $\mathbb{C}P^2$ is a stable bundle $B$ with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

**THEOREM:** (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **hyperkähler.**

**This theorem is proved using quivers**.

## ADHM construction

**DEFINITION:** Let $V$ and $W$ be complex vector spaces, with dimensions $c$ and $r$, respectively. The **ADHM data** is maps

$$A, B \in \mathrm{End}(V), I \in \mathrm{Hom}(W, V), J \in \mathrm{Hom}(V, W).$$

We say that ADHM data is

**stable**,

if there is no subspace $S \subsetneq V$ such that $A(S), B(S) \subset S$ and $I(W) \subset S$;

**costable**,

if there is no nontrivial subspace $S \subset V$ such that $A(S), B(S) \subset S$ and $S \subset \ker J$;

**regular**,

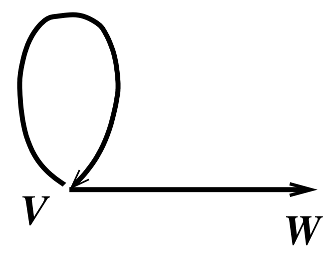
if it is both stable and costable.

**The ADHM equation** is $[A, B] + IJ = 0$.

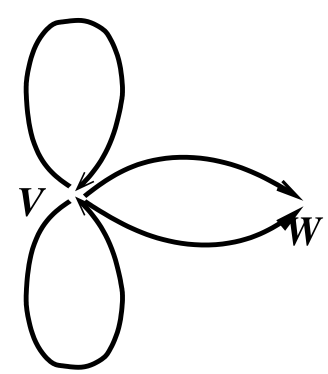
**THEOREM:** (Atiyah, Drinfeld, Hitchin, Manin) Framed rank $r$, charge $c$ instantons on $\mathbb{C}P^2$ are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on $\mathbb{C}P^2$ is identified with moduli of the corresponding quiver representation.**

## ADHM spaces as quiver varieties

Consider the quiver




The ADHM data is the set $Q$ of representations of the corresponding double quiver




The corresponding **holomorphic moment map is the ADHM equation** $A, B, I, J \longrightarrow [A, B] + IJ$ **with values in** $\mathrm{End}(V)$**.**

**The set of equivalence classes of ADHM solutions is** $Q /\!\!/\!\!/ U(V)$**.**

## Twistor space

**DEFINITION: Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if $M$ is compact, for generic $a, b, c$, $(M, L)$ has no divisors (Fujiki).

**DEFINITION:** A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

**EXAMPLE:** If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

**REMARK: For $M$ compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.**

## Hyperholomorphic connections

**REMARK:** Let $M$ be a hyperkähler manifold. **The group $SU(2)$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

**DEFINITION:** A **hyperholomorphic connection** on a vector bundle $B$ over $M$ is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

**REMARK:** Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda^{1,1}_I(M)$, **a hyperholomorphic connection defines a holomorphic structure on $B$** for each $I$ induced by quaternions.

**REMARK:** When $\dim_{\mathbb{H}} M = 1$, "hyperholomorphic" is synonymous with "anti-selfdual": $\Lambda^2(M)^{SU(2)} = \Lambda^-(M)$.

**THEOREM:** (Malgrange? Atiyah-Bott?) Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a (0,1)-part of connection $\nabla$ satisfying $(\Theta_\nabla)^{0,2} = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

## Twistor transform and hyperholomorphic bundles 1: direct twistor transform

**CLAIM:** Let $\sigma : \operatorname{Tw}(M) \longrightarrow M$ be the standard projection, where $M$ is hyperkähler and $\eta \in \Lambda^2 M$ a 2-form. Then $\sigma^*\eta$ **is a (1,1)-form iff** $\eta$ **is** $SU(2)$**-invariant.**

**COROLLARY:** Let $(B, \nabla)$ be a bundle with connection, and $\sigma^* B, \sigma^* \nabla$ its pullback to $\operatorname{Tw}(M)$. **Then** $(\sigma^* B, \sigma^* \nabla)$ **has (1,1)-curvature iff** $\nabla$ **has** $SU(2)$**-invariant curvature.**

**REMARK:** This construction produces a holomorphic vector bundle on $\operatorname{Tw}(M)$ starting from a connection with $SU(2)$-invariant curvature. It is called **direct twistor transform**. The **inverse twistor transform** produces a bundle with connection on $M$ from a holomorphic bundle on $\operatorname{Tw}(M)$.

**DEFINITION:** **A non-Hermitian hyperholomorphic connection** on a complex vector bundle $B$ is a connection (not necessarily Hermitian) which has $SU(2)$-invariant curvature.

## Twistor transform and hyperholomorphic bundles 2: inverse twistor transform

**DEFINITION:** Let $M$ be a hyperkähler manifold, and $\sigma : \operatorname{Tw}(M) \longrightarrow M$ its twistor space. For each point $x \in M$, $\sigma^{-1}(x)$ is a holomorphic rational curve in $\operatorname{Tw}(M)$. It is called **a horizontal twistor line**.

**THEOREM: (The inverse twistor transform; Kaledin-V.)** Let $B$ be a holomorphic vector bundle on $\operatorname{Tw}(M)$, which is trivial on any horizontal twistor line. Denote by $B_0$ the $C^\infty$-bundle on $M$ with fiber $H^0(B\big|_{\sigma^{-1}(x)})$ at $x \in M$. **Then $B_0$ admits a unique non-Hermitian hyperholomorphic connection $\nabla$** such that $B$ is isomorphic (as a holomorphic vector bundle) to its twistor transform $(\sigma^* B_0, (\sigma^*\nabla)^{0,1})$.

**REMARK:** The condition of being trivial on any horizontal twistor line is **open.** Therefore, **the moduli of holomorphic bundles on a $\operatorname{Tw}(M)$ contain an open subset corresponding to non-Hermitian hyperholomorphic connection on $M$.**

## Rational curves on twistor spaces

**DEFINITION:** Denote by $\text{Sec}(M)$ **the space of holomorphic sections** of the twistor fibration $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$. For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of $\pi$. The space of horizontal sections is denoted $\text{Sec}_{hor}(M) \subset \text{Sec}(M)$

**REMARK:** The space of horizontal sections of $\pi$ is identified with $M$. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of** $\text{Sec}_{hor}(M) \subset \text{Sec}(M)$ **is a smooth manifold of dimension** $2 \dim M$**.**

Let $B$ be a (Hermitian) hyperholomorphic bundle on $M$, and $W$ the deformation space of $B$, which is known to be hyperkähler. Denote by $\tilde{B}$ the holomorphic bundle on $\text{Tw}(M)$, obtained as a twistor transform of $B$. **Any deformation** $\tilde{B}_1$ **of** $\tilde{B}$ **gives a holomorphic map** $\mathbb{C}P^1 \longrightarrow \text{Tw}(W)$ mapping $L \in \mathbb{C}P^1$ to a bundle $\tilde{B}_1\big|_{(M,L)} \subset \text{Tw}(M)$, considered as a point in $(W, L)$.

**THEOREM:** (Kaledin-V.) This construction identifies **deformations of** $\tilde{B}$ **(with appropriate stability conditions) and rational curves** $S \in \text{Sec}(W)$**.** The twistor transforms of Hermitian hyperholomorphic bundles on $M$ correspond to $\text{Sec}_h(W) \subset \text{Sec}(W)$.

## Holomorphic bundles on $\mathbb{C}P^3$ and twistor sections

**DEFINITION: An instanton** on $\mathbb{C}P^2$ is a stable bundle $B$ with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

**THEOREM:** (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

**THEOREM:** (Kaledin-V.) This construction identifies **deformations of $\tilde{B}$ (with appropriate stability conditions) and rational curves $S \in \operatorname{Sec}(W)$.** The twistor transforms of Hermitian hyperholomorphic bundles on $M$ correspond to $\operatorname{Sec}_h(W) \subset \operatorname{Sec}(W)$.

**REMARK: There is a similar correspondence** between the holomorphic bundles on $\operatorname{Tw}(\mathbb{H}) = \mathbb{C}P^3 \backslash \mathbb{C}P^1$, with appropriate stability and framing conditions, and twistor sections in $\operatorname{Sec}(\mathcal{M}_{r,c})$.

## Mathematical instantons

**DEFINITION: A mathematical instanton** on $\mathbb{C}P^3$ is a stable bundle $B$ with $c_1(B)=0$ and $H^0(E(-1))=H^1(E(-2))=H^2(E(-2))=H^3(E(-3))=0$. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell=\mathbb{C}P^1\subset\mathbb{C}P^3$.

**DEFINITION: An instanton** on $\mathbb{C}P^2$ is a stable bundle $B$ with $c_1(B)=0$. **A framed instanton** is an instanton equipped with a trivialization $B|_x$ for some fixed point $x\in\mathbb{C}P^2$.

**THEOREM:** (Jardim–V.) The space $\mathbb{M}_{r,c}$ of framed mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections** Sec$(\mathcal{M}_{r,c})$**.**

**REMARK:** This correspondence is not surprising, if one realizes that $\mathrm{Tw}(\mathbb{H})=\mathbb{C}P^3\backslash\mathbb{C}P^1$.

**The space of instantons on $\mathbb{C}P^3$**

**THEOREM:** (Jardim–V.) **The space $\mathbb{M}_{r,c}$ is smooth.**

**REMARK:** To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_{r,c}$ is smooth, we develop **trihyperkähler reduction**, which is **a reduction defined on trisymplectic manifolds.**

We prove that $\mathbb{M}_{r,c}$ **is a trihyperkähler quotient** of a vector space by a reductive group action, hence smooth.

## Trisymplectic manifolds

**DEFINITION:** Let $\mathbf{\Omega}$ be a 3-dimensional space of holomorphic symplectic 2-forms on a complex manifold. Suppose that

- **$\mathbf{\Omega}$ contains a non-degenerate 2-form**
- For each non-zero degenerate $\Omega \in \mathbf{\Omega}$, one has $\operatorname{rk} \Omega = \frac{1}{2} \dim V$.

Then $\mathbf{\Omega}$ is called **a trisymplectic structure on $M$**.

**REMARK: The bundles $\ker \Omega$ are involutive,** because $\Omega$ is closed.

**THEOREM:** (Jardim–V.) For any trisymplectic structure on $M$, $M$ is equipped with a unique holomorphic, torsion-free connection, preserving the forms $\Omega_i$. It is called **the Chern connection** of $M$.

**REMARK: The Chern connection has holonomy in $Sp(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.**

**Trisymplectic structure on** $\operatorname{Sec}_0(M)$

**EXAMPLE:** Consider a hyperkähler manifold $M$. Let $I \in \mathbb{C}P^1$, and $ev_I : \operatorname{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Denote by $\Omega_I$ the holomorphic symplectic form on $(M, I)$. **Then $ev_I^* \Omega_I$, $I \in \mathbb{C}P^1$ generate a trisymplectic structure.**

**COROLLARY:** $\operatorname{Sec}_0(M)$ **is equipped with a holomorphic, torsion-free connection with holonomy in $Sp(n, \mathbb{C})$.**

## Trihyperkähler reduction

**DEFINITION: A trisymplectic moment map** $\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*$ takes vectors $\Omega \in \boldsymbol{\Omega}, g \in \mathfrak{g} = \text{Lie}(G)$ and maps them to a holomorphic function $f \in \mathcal{O}_M$, such that $df = \Omega \lrcorner g$, where $\Omega \lrcorner g$ denotes the contraction of $\Omega$ and the vector field $g$

**DEFINITION:** Let $(M, \boldsymbol{\Omega}, S_t)$ be a trisymplectic structure on a complex manifold $M$. Assume that $M$ is equipped with an action of a compact Lie group $G$ preserving $\boldsymbol{\Omega}$, and an equivariant trisymplectic moment map

$$\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*.$$

Let $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ be the corresponding **level set** of the moment map. Consider the action of the complex Lie group $G_{\mathbb{C}}$ on $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$. Assume that it is proper and free. Then the quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$ is a smooth manifold called **the trisymplectic quotient** of $(M, \boldsymbol{\Omega}, S_t)$, denoted by $M /\!\!/\!\!/\!\!/ G$.

**THEOREM:** Suppose that the restriction of $\boldsymbol{\Omega}$ to $\mathfrak{g} \subset TM$ is non-degenerate. **Then $M /\!\!/\!\!/\!\!/ G$ is trisymplectic.**

## Mathematical instantons and the twistor correspondence

**REMARK:** Using the monad description of mathematical instantons, **we prove that that the map $\mathrm{Sec}_0(\mathcal{M}_{r,c}) \longrightarrow \mathbb{M}_{r,c}$ to the space of mathematical instantons is an isomorphism** (Frenkel-Jardim, Jardim-V.).

**REMARK:** The smoothness of the space $\mathrm{Sec}_0(\mathcal{M}_{r,c}) = \mathbb{M}_{r,c}$ **follows from the trihyperkähler reduction procedure:**

**THEOREM:** Let $M$ be flat hyperkähler manifold, and $G$ a compact Lie group acting on $M$ by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient $M /\!\!/\!\!/ G$ is smooth. **Then there exists an open embedding**

$$\mathrm{Sec}_0(M) /\!\!/\!\!/\!\!/ G \xrightarrow{\Psi} \mathrm{Sec}_0(M /\!\!/\!\!/ G),$$

which is compatible with the trisymplectic structures on $\mathrm{Sec}_0(M) /\!\!/\!\!/\!\!/ G$ and $\mathrm{Sec}_0(M /\!\!/\!\!/ G)$.

**THEOREM:** If $M$ is the space of quiver representations which gives $M /\!\!/\!\!/ G = \mathcal{M}_{2,c}$, **$\Psi$ gives an isomorphism $\mathrm{Sec}_0(M) /\!\!/\!\!/\!\!/ G = \mathrm{Sec}_0(M /\!\!/\!\!/ G)$.**

## The 1-dimensional ADHM construction

**DEFINITION:** Let $V$ and $W$ be complex vector spaces, with dimensions $c$ and $r$, respectively. The **1-dimensional ADHM data** is maps

$$A_k, B_k \in \mathrm{End}(V), I_k \in \mathrm{Hom}(W,V), J_k \in \mathrm{Hom}(V,W), (k=0,1)$$

Choose homogeneous coordinates $[z_0 : z_1]$ on $\mathbb{C}P^1$ and define

$$\tilde{A} := A_0 \otimes z_0 + A_1 \otimes z_1 \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + B_1 \otimes z_1.$$

We say that 1-dimensional ADHM data is **globally regular** if $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$ is regular for every $p \in \mathbb{C}P^1$. The **1-dimensional ADHM equation** is $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p\tilde{J}_p = 0$, for all $p \in \mathbb{C}P^d$

**THEOREM:** (Marcos Jardim, Igor Frenkel) Let $C_1(r,c)$ denote the set of globally regular solutions of the 1-dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the 1-dimensional ADHM equations and isomorphism classes of rank $r$ instanton bundles** on $\mathbb{CP}^3$ framed at a fixed line $\ell$, where $\dim W = \mathrm{rk}(E)$ and $\dim V = c_2(E)$.

## 1-dimensional ADHM construction and the trisymplectic moment map

**THEOREM:** (Jardim, V.) Consider the natural (flat) trisymplectic structure on the space $U_{r,c}$ of 1-dimensional ADHM data, and let $\mu: U_{r,c} \longrightarrow H^0(\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \operatorname{End}(V))$ be a map associating to $C \in U_{r,c}$ and $p \in \mathbb{C}P^1$ the vector $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p \in \mathcal{O}_{\mathbb{C}P^1}(2) \otimes \operatorname{End}(V)$). **Then $\mu$ is a trisymplectic moment map. This identifies the set of equivalence classes of solutions of the 1-dimensional ADHM equation with the trihyperkähler quotient $U_{r,c} /\!\!/\!\!/\!\!/ U(V)$.**