Hyperkähler reduction and ALE spaces

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Moment maps

DEFINITION: (M,ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**.

CLAIM: An equivariant moment map exists whenever $H^1(G,\mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists. Further on, all moment maps will be tacitly considered equivariant.

Weinstein-Marsden theorem

DEFINITION: (Weinstein-Marsden) (M, ω) be a symplectic manifold, G a compact Lie group acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

Proof. Step 1: $T_x(\mu^{-1}(c)) = d\mu^{-1}(0)$, however, $d\mu$ is ω-dual to the space $\tau(\mathfrak{g})$ of vector fields tangent to the G-action, **hence $d\mu^{-1}(0) = \tau(\mathfrak{g})^\perp$.**

Step 2: Since μ is G-equivariant, G preserves $\mu^{-1}(c)$, hence $\tau(\mathfrak{g}) \subset d\mu^{-1}(0)$. This implies that **$\tau(\mathfrak{g}) \subset TM$ is isotropic** (that is, $\omega\big|_{\tau(\mathfrak{g})} = 0$.) Its ω-orthogonal complement in T_xM is $T_x(\mu^{-1}(c))$ (Step 1).

Step 3: Consider the **characteristic foliation** $\mathcal{F}$ on $\mu^{-1}(c)$, that is, the set of all $v \in T_x(\mu^{-1}(c))$ such that $\omega(v, w) = 0$ for all $w \in T_x(\mu^{-1}(c))$ From Step 2 **we obtain that $\mathcal{F} = \tau(\mathfrak{g})$.**

Step 4: Since $\omega\big|_{\mu^{-1}(c)}$ is closed, it satisfies $\operatorname{Lie}_v(\omega) = 0$ for all $v \in \mathcal{F}$. This implies that it is lifted from the leaf space of characteristic foliation, identified with $M/\!\!/G$. ■

Symplectic reduction and GIT

THEOREM: Let (M, I, ω) be a Kähler manifold, $G_{\mathbb{C}}$ a complex reductive Lie group acting on M by holomorphic automorphisms, and G its compact form acting isometrically. **Then $M/\!\!/G$ is a Kähler manifold.**

Proof: Since the orbits of the $G_{\mathbb{C}}$-action are complex subvarieties, they are symplectic. Since the orbits of $G \subset G_{\mathbb{C}}$ are isotropic, and their dimension is half of dimension of orbits of $G_{\mathbb{C}}$, they are actially Lagrangian subvarieties in orbits of $G_{\mathbb{C}}$. Therefore, $\mu^{-1}(c)$ intersects each orbit of $G_{\mathbb{C}}$ in a G-orbit. **We have identified $M/\!\!/G$ with a space of $G_{\mathbb{C}}$-orbits which intersect $\mu^{-1}(c)$.** ■

REMARK: In such a situation, $M/\!\!/G$ is called **the Kähler quotient**, or **GIT quotient**. The choice of a central element $c \in \mathfrak{g}^*$ is known as a choice of **stability data**.

REMARK: The points of $M/\!\!/G$ are in bijective correspondence with the orbits of $G_{\mathbb{C}}$ which intersect $\mu^{-1}(c)$. Such orbits are called **polystable**, and the intersection of a $G_{\mathbb{C}}$-orbit with $\mu^{-1}(c)$ **is a G-orbit.**

Kähler reduction and a Kähler potential

DEFINITION: Kähler potential on a Kähler manifold (M, ω) is a function ψ such that $dd^c\psi = \omega$.

PROPOSITION: Let G be a real Lie group acting on a Kähler manifold M by holomorphic isometries, and ψ be a G-invariant Kähler potential. **Then the moment map $\mathfrak{g} \times M \xrightarrow{\mu_g} \mathbb{R}$ can be written as $g, m \longrightarrow -\operatorname{Lie}_{Iv} \psi$,** where $v = \tau(g) \in TM$ is the tangent vector field associated with $g \in \mathfrak{g}$.

Proof: Since ψ is G-invariant, and I is G-invariant, we have $0 = \operatorname{Lie}_v d^c\psi = (dd^c\psi) \lrcorner v + d(\langle d^c\psi, v\rangle)$. Using $\omega = dd^c\psi$, we rewrite this equation as $\omega \lrcorner v = -d(\langle d^c\psi, v\rangle)$, giving an equation for the moment map $\mu_g = -\langle d^c\psi, v\rangle$. Acting by I on both sides, we obtain $\mu_g = -\langle d\psi, Iv\rangle = -\operatorname{Lie}_{Iv} \psi$. ■

COROLLARY: Let V be a Hermitian representation of a compact Lie group G. **Then the corresponding moment map can be written as $\mu_g(v) = -\operatorname{Lie}_{Ig} |v|^2 = -\frac{1}{2}\langle v, Ig(v)\rangle$.** ■

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

REMARK:
The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is **holomorphic and symplectic** on (M, I).

Hyperkähler reduction

DEFINITION: Let G be a compact Lie group, ρ its action on a hyperkähler manifold M by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g \rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one three Kähler forms associated with the hyperkähler structure.

DEFINITION: Let ξ_1, ξ_2, ξ_3 be three G-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)
The quotient $M /\!\!/\!\!/ G$ is hyperkaehler.

Holomorphic moment map

Let $\Omega := \omega_J + \sqrt{-1}\,\omega_K$. **This is a holomorphic symplectic (2,0)-form on (M, I).**

The proof of HKLR theorem. Step 1: Let μ_J, μ_K be the moment map associated with ω_J, ω_K, and $\mu_{\mathbb{C}} := \mu_J + \sqrt{-1}\,\mu_K$. Then $\langle d\mu_{\mathbb{C}}, g\rangle = i_{\rho_g}(\Omega)$. Therefore, $d\mu_{\mathbb{C}} \in \Lambda^{1,0}(M, I) \otimes \mathfrak{g}^*$.

Step 2: This implies that the map $\mu_{\mathbb{C}}$ is holomorphic. It is called **the holomorphic moment map.**

Step 3: By definition, $M/\!\!/\!\!/G = \mu_{\mathbb{C}}^{-1}(c)/\!\!/G$, where $c \in \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ is a central element. **This is a Kähler manifold,** because it is a Kähler quotient of a Kähler manifold.

Step 4: We obtain 3 complex structures I, J, K on the hyperkähler quotient $M/\!\!/\!\!/G$. **They are compatible in the usual way** (an easy exercise). ■

Quiver representations

DEFINITION: A **quiver** is an oriented graph. A **quiver representation** is a diagram of complex Hermitian vector spaces and arrows associated with a quiver:

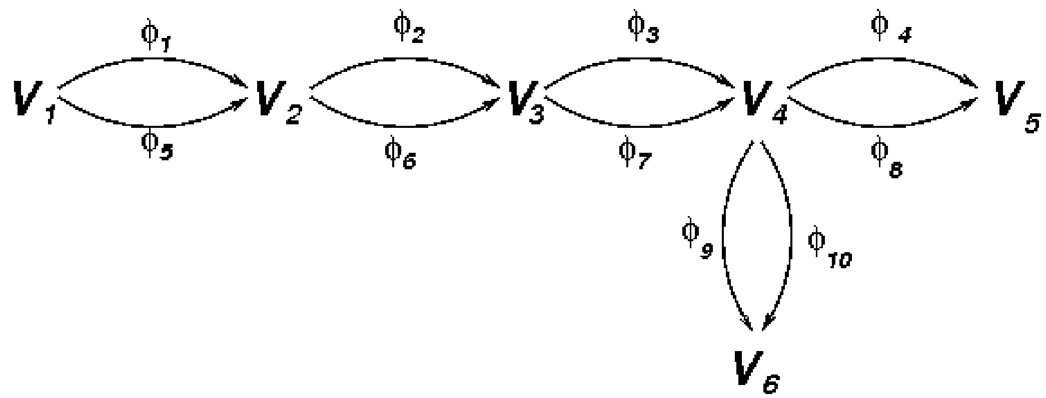
$$\begin{tikzcd}
V_1 \arrow[r, "\phi_1"] & V_2 \arrow[r, "\phi_2"] & V_3 \arrow[r, "\phi_3"] & V_4 \arrow[r, "\phi_4"] \arrow[d, "\phi_9"'] & V_5 \\
& & & V_6 &
\end{tikzcd}$$

Here, V_i are vector spaces, and φ_i linear maps.

REMARK: If one fixes the spaces V_i, the space of quiver representations is a Hermitian vector space.

Quiver varieties

Starting from a single graph, one can double it up, as follows, obtaining a Nakajima double quiver.



A Nakajima quiver for the Dynkin diagram D_5.

CLAIM: The space M of representations of a Nakajima's double quiver is a quaternionic vector space, and the group $G := U(V_1) \times U(V_2) \times ... \times U(V_n)$ acts on M preserving the quaternionic structure.

DEFINITION: A **Nakajima quiver variety** is a quotient $M /\!\!/\!\!/ G$.

Hyperkähler manifolds as quiver varieties

Many non-compact hyperkähler manifolds are obtained as quiver varieties.

EXAMPLE: A 4-dimensional ALE **(asymptotically locally Euclidean)** space obtained as a resolution of **a du Val singularity**, that is, a quotient $\mathbb{C}^2/G$, where $G \subset SU(2)$ is a finite group.

REMARK: Since finite subgroups of $SU(2)$ are classified by the Dynkin diagrams of type A,D,E, these ALE quotients are called **ALE spaces of A-D-E type**.

DEFINITION: A metric on a complete manifold M is **asymptotically flat** if its curvature satisfies $|R| = O(r^{-3})$, where r is a distance from a point.

EXAMPLE: Any ALE space M **admit an asymptotically flat hyperkähler metric.** Moreover, all asymptotically flat hyperkähler metrics on M **are obtained through the Nakajima quiver constructuion** (Kronheimer, later interpreted by Nakajima)

EXAMPLE: The moduli asymptotically flat Hermitian Yang-Mills connections on ALE spaces is a complete hyperkähler space, **also obtained through the Nakajima quiver constructuion** (Kronheimer, Nakajima).

Quaternion action on $P = Q \otimes_{\mathbb{R}} \mathfrak{u}(R)$

Let $\Gamma \subset SU(2)$ be a finite subgroup; a posteriori, Γ is of ADE type, that is, cyclic, dihedral, or a group of symmetries of a Platonic body in $\mathbb{R}^3$. Denote by Q is fundamental 2-dimensional representation, and by R its regular representation, $R = \mathbb{C}[\Gamma]$. Let $P := Q \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(R)$ and $M := P^\Gamma$ be the space of Γ-invariant vectors in P.

CLAIM: Choose a Γ-invariant Hermitian structure on R such that $\operatorname{End}(R) = \mathfrak{u}(R) \otimes_{\mathbb{R}} \mathbb{C}$. **Then the space $P = Q \otimes_{\mathbb{R}} \mathfrak{u}(R)$ is equipped with a natural Γ-invariant $\mathbb{H}$-action.**

Proof: Q is $\mathbb{C}^2$, real and imaginary parts of the $\mathbb{C}$-linear symplectic form define a Γ-invariant $\mathbb{H}$-action on Q, hence $P = Q \otimes_{\mathbb{R}} \mathfrak{u}(R)$ is a quaternionic vector space, with $\Gamma \subset \mathfrak{u}(P, \mathbb{H})$. ∎

ALE spaces as quiver varieties

PROPOSITION: Denote by $F \subset U(R)$ the centralizer of Γ acting on R. Let $X := M /\!\!/\!\!/ U(R) := \frac{\mu^{-1}(0)}{F}$ be the result of hyperkähler reduction, associated with the $0 \in (\mathfrak{f}^*)^3$. **Then $X = \frac{\mathbb{C}^2}{\Gamma}$.**

Proof: Soon. ■

REMARK: This result gives a combinatorial description of $X_r := \frac{\mu_{\mathbb{C}}^{-1}(r)}{F}$ which is understood as a smooth deformation of $\frac{\mathbb{C}^2}{\Gamma}$. **This puts a hyperkähler metric on X.** A standard argument ("symplectic blow-up") implies that **the space X is a holomorphic symplectic resolution of singularities of $\frac{\mathbb{C}^2}{\Gamma}$.**

REMARK: Consider an extended Dynkin diagram for the A, D, E root system as a graph. **Using an appropriate set of vector spaces, we obtain M as a quiver space for these graphs.**

Hyperkähler reduction of $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathfrak{u}(R)$.

Claim 1: Consider an action of $U(R)$ on $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathfrak{u}(R)^2$. Then **the moment map $\mu_{\mathbb{C}}$ can be written as $\mu_{\mathbb{C}}(\alpha, \beta) = [\alpha, \alpha^*] + [\beta, \beta^*] + \sqrt{-1}\,[\alpha, \beta]$.** Moreover, $\mu^{-1}(0)$ **is the set of all $(\alpha, \beta) \in \mathfrak{gl}(R)^2$ which can be simultaneously diagonalized in the same orthonormal basis.**

Proof. Step 1: The formula for the moment map follows because the Kähler potential on $\mathfrak{gl}(R)$ is $A \mapsto \operatorname{Tr}(AA^*)$.

Step 2: For any $(\alpha, \beta) \in \mu_{\mathbb{C}}^{-1}(0)$, we have $[\alpha, \beta] = [\alpha, \alpha^*] + [\beta, \beta^*] = 0$. This gives $[\operatorname{ad}_{\alpha^*} \operatorname{ad}_\alpha + \operatorname{ad}_{\beta^*} \operatorname{ad}_\beta](\alpha^*) = 0$ Since $\operatorname{ad}_{\alpha^*} \operatorname{ad}_\alpha$ and $\operatorname{ad}_{\beta^*} \operatorname{ad}_\beta$ are positive operators, we obtain $[\alpha, \alpha^*] = 0$, hence $[\beta, \beta^*] = 0$ and α and β can be both diagonalized in the same orthonormal basis. ■

Hyperkähler reduction of $M = Q \otimes_{\mathbb{R}} \mathfrak{u}(R)^\Gamma$.

CLAIM: Let $\{e_\gamma, \gamma \in \Gamma\}$ be the standard basis in $R = \mathbb{C}[\Gamma]$, and $L \subset M$ the set of all pairs $(\alpha, \beta) \in \mathfrak{u}(R)^2 = P$ such that α and β are diagonal in this basis. Denote by $\mu_{\mathbb{C}} : P \longrightarrow \mathfrak{f} \otimes_{\mathbb{R}} \mathbb{C}$ the complex moment map. **Then $L \cap \mu_{\mathbb{C}}^{-1}(0)$ is an orbit of Γ.**

Proof. Step 1: By Claim 1, **α and β can be both diagonalized in the same orthonormal basis.**

Step 2: Take a Γ-invariant orthonormal basis in R. Since F acts on the space of such bases freely and transitively, the intersection $L \cap \mu_{\mathbb{C}}^{-1}(0)$ contains a pair of matrices which are diagonal in the basis $e_1, e_{\gamma_1}, ...$ Since Γ acts transitively on $e_1, e_{\gamma_1}, ...$, the matrices α and β are uniquely determined by the coefficients a, b in $\alpha(e_1) = ae_1$ and $\beta(e_1) = be_1$. The group Γ acts on the pairs (a, b) mapping e_1 to e_γ, and the coefficients (a, b) are transformed as points in Q.

Step 3: Two points of L lie in the same orbit of F if and only if they lie in the same orbit of Γ. ■

COROLLARY: $\frac{\mu_{\mathbb{C}}^{-1}(0)}{F} = \frac{\mathbb{C}}{\Gamma}$.