Special Holonomy and the ADHM Construction

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Plan

0. GIT, symplectic reduction, hyperkähler reduction

1. Quiver varieties and ADHM construction

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Moment maps

DEFINITION: (M,ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**.

DEFINITION: A G-invariant $c \in \mathfrak{g}^*$ is called **central**.

CLAIM: An equivariant moment map exists whenever $H^1(G,\mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists.

Symplectic reduction and GIT

DEFINITION: (Weinstein-Marsden) (M, ω) be a symplectic manifold, G a compact Lie group acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

THEOREM: Let (M, I, ω) be a Kähler manifold, $G_{\mathbb{C}}$ a complex reductive Lie group acting on M by holomorphic automorphisms, and G is compact form acting isometrically. **Then $M/\!\!/G$ is a Kähler orbifold.**

REMARK: In such a situation, $M/\!\!/G$ is called **the Kähler quotient**, or **GIT quotient**. The choice of a central element $c \in \mathfrak{g}^*$ is known as a choice of **stability data**.

REMARK: The points of $M/\!\!/G$ are in bijective correspondence with the orbits of $G_{\mathbb{C}}$ which intersect $\mu^{-1}(c)$. Such orbits are called **polystable**, and the intersection of a $G_{\mathbb{C}}$-orbit with $\mu^{-1}(c)$ is a G-orbit.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

REMARK:
The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is **holomorphic and symplectic** on (M, I).

Hyperkähler reduction

DEFINITION: Let G be a compact Lie group, ρ its action on a hyperkähler manifold M by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g \rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one three Kähler forms associated with the hyperkähler structure.

DEFINITION: Let ξ_1, ξ_2, ξ_3 be three G-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)
The quotient $M /\!\!/\!\!/ G$ is hyperkaehler.

Quiver representations

DEFINITION: A **quiver** is an oriented graph. A **quiver representation** is a diagram of complex Hermitian vector spaces and arrows associated with a quiver:

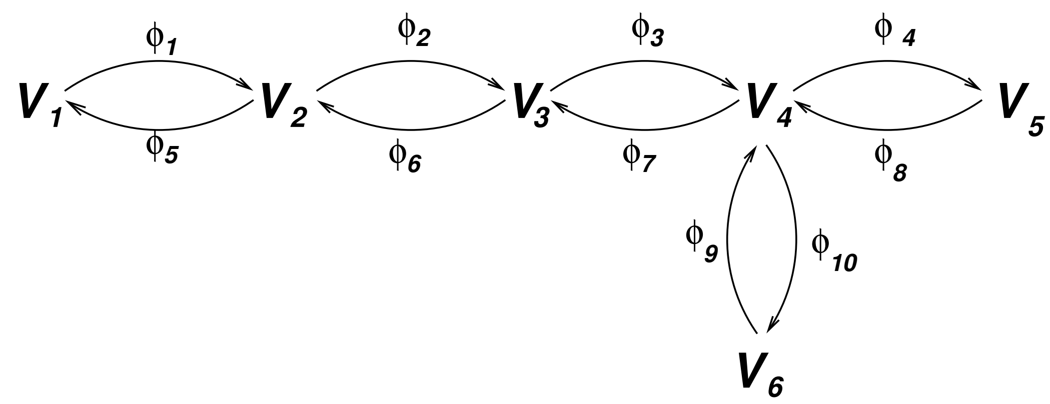
$$\begin{tikzcd} V_1 \arrow[r, "\phi_1"] & V_2 \arrow[r, "\phi_2"] & V_3 \arrow[r, "\phi_3"] & V_4 \arrow[r, "\phi_4"] \arrow[d, "\phi_9"'] & V_5 \\ & & & V_6 & \end{tikzcd}$$

Here, V_i are vector spaces, and φ_i linear maps.

REMARK: If one fixes the spaces V_i, the space of quiver representations is a Hermitian vector space.

Quiver varieties

Starting from a single graph, one can double it up, as follows, obtaining a Nakajima double quiver.



A Nakajima quiver for the Dynkin diagram D_5.

CLAIM: The space M of representations of a Nakajima's double quiver is a quaternionic vector space, and the group $G := U(V_1) \times U(V_2) \times ... \times U(V_n)$ acts on M preserving the quaternionic structure.

DEFINITION: A **Nakajima quiver variety** is a quotient $M/\!/\!/G$.

Hyperkähler manifolds as quiver varieties

Many non-compact hyperkähler manifolds are obtained as quiver varieties.

EXAMPLE: A 4-dimensional ALE **(asymptotically locally Euclidean)** space obtained as a resolution of **a du Val singularity**, that is, a quotient $\mathbb{C}^2/G$, where $G \subset SU(2)$ is a finite group.

REMARK: Since finite subgroups of $SU(2)$ are classified by the Dynkin diagrams of type A,D,E, these ALE quotients are called **ALE spaces of A-D-E type**.

EXAMPLE: The moduli asymptotically flat Hermitian Yang-Mills connections on ALE spaces of A-D-E type.

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

THEOREM: (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **hyperkähler.**

This theorem is proved using quivers.

ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **ADHM data** is maps

$$A, B \in \mathrm{End}(V), I \in \mathrm{Hom}(W, V), J \in \mathrm{Hom}(V, W).$$

We say that ADHM data is

stable,

if there is no subspace $S \subsetneq V$ such that $A(S), B(S) \subset S$ and $I(W) \subset S$;

costable,

if there is no nontrivial subspace $S \subset V$ such that $A(S), B(S) \subset S$ and $S \subset \ker J$;

regular,

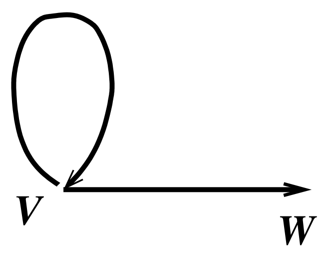
if it is both stable and costable.

The ADHM equation is $[A, B] + IJ = 0$.

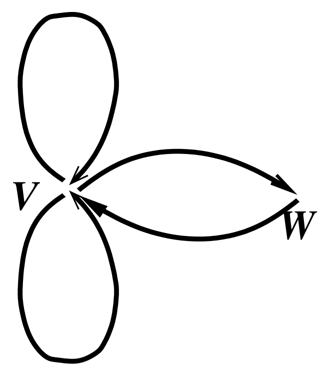
THEOREM: (Atiyah, Drinfeld, Hitchin, Manin) Framed rank r, charge c instantons on $\mathbb{C}P^2$ are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on $\mathbb{C}P^2$ is identified with moduli of the corresponding quiver representation.**

ADHM spaces as quiver varieties

Consider the quiver



The ADHM data is the set Q of representations of the corresponding double quiver



The corresponding **holomorphic moment map is the ADHM equation** $A, B, I, J \longrightarrow [A, B] + IJ$ **with values in** $\mathrm{End}(V)$**.**

The set of equivalence classes of ADHM solutions is $Q /\!\!/\!\!/ U(V)$**.**

Twistor space

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Hyperholomorphic connections

REMARK: Let M be a hyperkähler manifold. **The group $SU(2)$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda^{1,1}_I(M)$, **a hyperholomorphic connection defines a holomorphic structure on B** for each I induced by quaternions.

REMARK: When $\dim_{\mathbb{H}} M = 1$, "hyperholomorphic" is synonymous with "anti-selfdual": $\Lambda^2(M)^{SU(2)} = \Lambda^-(M)$.

THEOREM: Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a (0,1)-part of connection ∇ satisfying $(\Theta_\nabla)^{0,2} = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

Twistor transform and hyperholomorphic bundles 1: direct twistor transform

CLAIM: Let $\sigma : \operatorname{Tw}(M) \longrightarrow M$ be the standard projection, where M is hyperkähler and $\eta \in \Lambda^2 M$ a 2-form. Then **$\sigma^*\eta$ is a (1,1)-form iff η is $SU(2)$-invariant.**

COROLLARY: Let (B, ∇) be a bundle with connection, and $\sigma^*B, \sigma^*\nabla$ its pullback to $\operatorname{Tw}(M)$. **Then $(\sigma^*B, \sigma^*\nabla)$ has (1,1)-curvature iff ∇ has $SU(2)$-invariant curvature.**

REMARK: This construction produces a holomorphic vector bundle on $\operatorname{Tw}(M)$ starting from a connection with $SU(2)$-invariant curvature. It is called **direct twistor transform**. The **inverse twistor transform** produces a bundle with connection on M from a holomorphic bundle on $\operatorname{Tw}(M)$.

DEFINITION: A non-Hermitian hyperholomorphic connection on a complex vector bundle B is a connection (not necessarily Hermitian) which has $SU(2)$-invariant curvature.

Twistor transform and hyperholomorphic bundles 2: inverse twistor transform

DEFINITION: Let M be a hyperkähler manifold, and $\sigma : \text{Tw}(M) \longrightarrow M$ its twistor space. For each point $x \in M$, $\sigma^{-1}(x)$ is a holomorphic rational curve in $\text{Tw}(M)$. It is called **a horizontal twistor section**.

THEOREM: (The inverse twistor transform; Kaledin-V.) Let B be a holomorphic vector bundle on $\text{Tw}(M)$, which is trivial on any horizontal twistor line. Denote by B_0 the C^∞-bundle on M with fiber $H^0(B\big|_{\sigma^{-1}(x)})$ at $x \in M$. **Then B_0 admits a unique non-Hermitian hyperholomorphic connection ∇** such that B is isomorphic (as a holomorphic vector bundle) to its twistor transform $(\sigma^* B_0, (\sigma^*\nabla)^{0,1})$.

REMARK: The condition of being trivial on any horizontal twistor line is **open.** Therefore, **the moduli of holomorphic bundles on a $\text{Tw}(M)$ contain an open subset corresponding to non-Hermitian hyperholomorphic connection on M.**

Rational curves on twistor spaces

DEFINITION: Denote by $\operatorname{Sec}(M)$ **the space of holomorphic sections** of the twistor fibration $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$. For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of π. The space of horizontal sections is denoted $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$ is a smooth manifold of dimension $2\dim M$.**

Let B be a (Hermitian) hyperholomorphic bundle on M, and W the deformation space of B, which is known to be hyperkähler. Denote by $\tilde{B}$ the holomorphic bundle on $\operatorname{Tw}(M)$, obtained as a twistor transform of B. **Any deformation $\tilde{B}_1$ of $\tilde{B}$ gives a holomorphic map $\mathbb{C}P^1 \longrightarrow \operatorname{Tw}(W)$** mapping $L \in \mathbb{C}P^1$ to a bundle $\tilde{B}_1\big|_{(M,L)} \subset \operatorname{Tw}(M)$, considered as a point in (W, L).

THEOREM: (Kaledin-V.) This construction identifies **deformations of $\tilde{B}$ (with appropriate stability conditions) and rational curves $S \in \operatorname{Sec}(W)$.** The twistor transforms of Hermitian hyperholomorphic bundles on M correspond to $\operatorname{Sec}_h(W) \subset \operatorname{Sec}(W)$.

Holomorphic bundles on $\mathbb{C}P^3$ and twistor sections

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

THEOREM: (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

REMARK: There is a similar correspondence between the holomorphic bundles on $\operatorname{Tw}(\mathbb{H}) = \mathbb{C}P^3 \backslash \mathbb{C}P^1$, with appropriate stability and framing conditions, and twistor sections in $\operatorname{Sec}(\mathcal{M}_{r,c})$.

Mathematical instantons

DEFINITION: A mathematical instanton on $\mathbb{C}P^3$ is a stable bundle B with $c_1(B)=0$ and $H^0(E(-1))=H^1(E(-2))=H^2(E(-2))=H^3(E(-3))=0$. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell=\mathbb{C}P^1\subset\mathbb{C}P^3$.

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B)=0$. **A framed instanton** is an instanton equipped with a trivialization $B|_x$ for some fixed point $x\in\mathbb{C}P^2$.

THEOREM: (Jardim–V.) The space $\mathbb{M}_{r,c}$ of framed mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections** Sec$(\mathcal{M}_{r,c})$**.**

REMARK: This correspondence is not surprising, if one realizes that $\operatorname{Tw}(\mathbb{H})=\mathbb{C}P^3\backslash\mathbb{C}P^1$.

The space of instantons on $\mathbb{C}P^3$

THEOREM: (Jardim–V.) **The space $\mathbb{M}_{r,c}$ is smooth.**

REMARK: To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_{r,c}$ is smooth, we develop **trihyperkähler reduction**, which is **a reduction defined on trisymplectic manifolds.**

We prove that $\mathbb{M}_{r,c}$ **is a trihyperkähler quotient** of a vector space by a reductive group action, hence smooth.

Trisymplectic manifolds

DEFINITION: Let $\mathbf{\Omega}$ be a 3-dimensional space of holomorphic symplectic 2-forms on a complex manifold. Suppose that

- **$\mathbf{\Omega}$ contains a non-degenerate 2-form**
- For each non-zero degenerate $\Omega \in \mathbf{\Omega}$, one has $\operatorname{rk} \Omega = \frac{1}{2} \dim V$.

Then $\mathbf{\Omega}$ is called **a trisymplectic structure on M**.

REMARK: The bundles $\ker \Omega$ are involutive, because Ω is closed.

THEOREM: (Jardim–V.) For any trisymplectic structure on M, M is equipped with a unique holomorphic, torsion-free connection, preserving the forms Ω_i. It is called **the Chern connection** of M.

REMARK: The Chern connection has holonomy in $Sp(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

Trisymplectic structure on $\operatorname{Sec}_0(M)$

EXAMPLE: Consider a hyperkähler manifold M. Let $I \in \mathbb{C}P^1$, and $ev_I : \operatorname{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \operatorname{Sec}_0(M)$ to $S(I)$. Denote by Ω_I the holomorphic symplectic form on (M, I). **Then $ev_I^*\Omega_I$, $I \in \mathbb{C}P^1$ generate a trisymplectic structure.**

COROLLARY: $\operatorname{Sec}_0(M)$ **is equipped with a holomorphic, torsion-free connection with holonomy in $Sp(n, \mathbb{C})$.**

Trihyperkähler reduction

DEFINITION: A trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*$ takes vectors $\Omega \in \boldsymbol{\Omega}, g \in \mathfrak{g} = \operatorname{Lie}(G)$ and maps them to a holomorphic function $f \in \mathcal{O}_M$, such that $df = \Omega \lrcorner g$, where $\Omega \lrcorner g$ denotes the contraction of Ω and the vector field g

DEFINITION: Let $(M, \boldsymbol{\Omega}, S_t)$ be a trisymplectic structure on a complex manifold M. Assume that M is equipped with an action of a compact Lie group G preserving $\boldsymbol{\Omega}$, and an equivariant trisymplectic moment map

$$\boldsymbol{\mu}_{\mathbb{C}}: M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \boldsymbol{\Omega}^*.$$

Let $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ be the corresponding **level set** of the moment map. Consider the action of the complex Lie group $G_{\mathbb{C}}$ on $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$. Assume that it is proper and free. Then the quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$ is a smooth manifold called **the trisymplectic quotient** of $(M, \boldsymbol{\Omega}, S_t)$, denoted by $M /\!\!/\!\!/\!\!/ G$.

THEOREM: Suppose that the restriction of $\boldsymbol{\Omega}$ to $\mathfrak{g} \subset TM$ is non-degenerate. **Then $M /\!\!/\!\!/\!\!/ G$ is trisymplectic.**

Mathematical instantons and the twistor correspondence

REMARK: Using the monad description of mathematical instantons, **we prove that that the map $\mathrm{Sec}_0(\mathcal{M}_{r,c}) \longrightarrow \mathbb{M}_{r,c}$ to the space of mathematical instantons is an isomorphism** (Frenkel-Jardim, Jardim-V.).

REMARK: The smoothness of the space $\mathrm{Sec}_0(\mathcal{M}_{r,c}) = \mathbb{M}_{r,c}$ **follows from the trihyperkähler reduction procedure:**

THEOREM: Let M be flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient $M/\!\!/\!\!/G$ is smooth. **Then there exists an open embedding**

$$\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G \xrightarrow{\Psi} \mathrm{Sec}_0(M/\!\!/\!\!/G),$$

which is compatible with the trisymplectic structures on $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G$ and $\mathrm{Sec}_0(M/\!\!/\!\!/G)$.

THEOREM: If M is the space of quiver representations which gives $M/\!\!/\!\!/G = \mathcal{M}_{2,c}$, **$\Psi$ gives an isomorphism $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G = \mathrm{Sec}_0(M/\!\!/\!\!/G)$.**

The 1-dimensional ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **1-dimensional ADHM data** is maps

$$A_k, B_k \in \mathrm{End}(V), I_k \in \mathrm{Hom}(W,V), J_k \in \mathrm{Hom}(V,W), (k=0,1)$$

Choose homogeneous coordinates $[z_0 : z_1]$ on $\mathbb{C}P^1$ and define

$$\tilde{A} := A_0 \otimes z_0 + A_1 \otimes z_1 \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + B_1 \otimes z_1.$$

We say that 1-dimensional ADHM data is **globally regular** if $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$ is regular for every $p \in \mathbb{C}P^1$. The **1-dimensional ADHM equation** is $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p = 0$, for all $p \in \mathbb{C}P^d$

THEOREM: (Marcos Jardim, Igor Frenkel) Let $C_1(r,c)$ denote the set of globally regular solutions of the 1-dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the 1-dimensional ADHM equations and isomorphism classes of rank r instanton bundles** on $\mathbb{CP}^3$ framed at a fixed line ℓ, where $\dim W = \mathrm{rk}(E)$ and $\dim V = c_2(E)$.

1-dimensional ADHM construction and the trisymplectic moment map

THEOREM: (Jardim, V.) Consider the natural (flat) trisymplectic structure on the space $U_{r,c}$ of 1-dimensional ADHM data, and let $\mu : U_{r,c} \longrightarrow H^0(\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \operatorname{End}(V))$ be a map associating to $C \in U_{r,c}$ and $p \in \mathbb{C}P^1$ the vector $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p \in \mathcal{O}_{\mathbb{C}P^1}(2) \otimes \operatorname{End}(V)$). **Then μ is a trisymplectic moment map. This identifies the set of equivalence classes of solutions of the 1-dimensional ADHM equation with the trihyperkähler quotient $U_{r,c} /\!\!/\!\!/\!\!/ U(V)$.**