# The Ahlfors current

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## Schwartz lemma

**CLAIM: (maximum principle)** Let $f$ be a holomorphic function defined on an open set $U$. **Then $f$ cannot have strict maxima in $U$. If $f$ has non-strict maxima, it is constant.**

**EXERCISE: Prove the maximum principle.**

**LEMMA: (Schwartz lemma)** Let $f: \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

**Proof:** Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = const$. ■

## Conformal automorphisms of the disk

**CLAIM:** Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group $\operatorname{Aut}(\Delta)$ of its holomorphic automorphisms acts on $\Delta$ transitively.**

**Proof:** Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$, which is implied from

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1,$$

when $|z| = 1$. ■

**REMARK:** The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form $h$ of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \ | \ h(l,l) > 0\}$ by holomorphic automorphisms.

**COROLLARY:** Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi : PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then $\Psi$ is an isomorphism.**

**COROLLARY:** Let $h$ be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then $(\Delta, h)$ is conformally equivalent to $(\Delta,$ flat metric$)$.**

## Upper half-plane

**REMARK:** The map $z \longrightarrow -\sqrt{-1}\,(z-1)^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$.

**PROPOSITION: The group** $\operatorname{Aut}(\Delta)$ **acts on the upper half-plane** $\mathbb{H}$ **as** $z \xrightarrow{A} \frac{az+b}{cz+d}$**, where** $a, b, c, d \in \mathbb{R}$**, and** $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$**.**

**REMARK:** The group of such $A$ is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$. Since $PSL(2,\mathbb{R})$ acts on its Lie algebra preserving the Killing form, $PSL(2,\mathbb{R})$ embeds to $SO(1,2)$. Both of these groups are 3-dimensional, since they are isomorphic.

**REMARK:** We have shown that $\mathbb{H} = SO(1,2)/S^1$. This gives a **natural isomorphism of $\mathbb{H}$ and the hyperbolic space.** Under this isomorphism, **holomorphic automorphisms correspond to isometries.**

## Poincaré metric on disk

**DEFINITION: Poincaré metric** on a unit disk $\Delta \subset \mathbb{C}$ is an Aut($\Delta$)-invariant metric **(it is unique up to a constant multiplier)**.

**DEFINITION:** Let $f : M \longrightarrow M_1$ be a map of metric spaces. Then $f$ is called **$C$-Lipschitz** if $d(x, y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is $C$-Lipschitz for some $C > 0$.

**THEOREM: (Schwartz-Pick lemma)**
**Any holomorphic map $\varphi : \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poincaré metric.**

**Proof. Step 1:** We need to prove that for each $x \in \Delta$ the norm of the differential satisfies $|D\varphi_x| \leqslant 1$. Since the automorphism group acts on $\Delta$ transitively, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

**Step 2:** This is Schwartz lemma. ■

## Kobayashi pseudometric

**DEFINITION: Pseudometric** on $M$ is a function $d: M\times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y)=d(y,x)$ and satisfies the triangle inequality $d(x,y)+d(y,z)\geqslant d(x,z)$.

**REMARK:** Let $\mathfrak{D}$ be a set of pseudometrics. **Then $d_{\max}(x,y):=\sup_{d\in\mathfrak{D}} d(x,y)$ is also a pseudometric.**

**DEFINITION:** The **Kobayashi pseudometric** on a complex manifold $M$ is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to $M$ is distance-decreasing.

**EXERCISE:** Prove that **the distance between points $x,y$ in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting $x$ to $y$.**

**EXAMPLE:** The Kobayashi pseudometric on $\mathbb{C}$ vanishes.

**CLAIM: Any holomorphic map $X \stackrel{\varphi}{\longrightarrow} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.**

**Proof:** If $x\in X$ is connected to $x'$ by a sequence of Poincare disks $\Delta_1,...,\Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1),...,\varphi(\Delta_n)$. ■

## Kobayashi hyperbolic manifolds

**COROLLARY:** Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since $x_i, y_i$ belongs to $\Delta$, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then $d_K(x,y) \geqslant \max_i d_P(x_i, y_i)$.**

**Proof:** Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipschitz. ■

**DEFINITION:** A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric $d_K$ is non-degenerate.

**DEFINITION:** A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

**COROLLARY: Any bounded domain $\Omega$ in $\mathbb{C}^n$ is Kobayashi hyperbolic.**

**Proof:** Without restricting generality, we may assume that $\Omega \subset B$ where $B$ is an open ball. Then the Kobayashi distance in $\Omega$ is $\geqslant$ that in $B$. However, the Kobayashi distance in $B$ is bounded by the metric $d(x,y) := \max_i d_P(x_i, y_i)$ as follows from above. ■

## Brody curves and Brody maps

**DEFINITION:** Let $M$ be a complex Hermitian manifold. **Brody curve** is a non-constant holomorphic map $f: \mathbb{C} \longrightarrow M$ such that $|df| \leqslant C$ for some constant $C$. Here $|df|$ is understood as an operator norm of $df: T_z\mathbb{C} \longrightarrow TM$, where $\mathbb{C}$ is equipped with the standard Euclidean metric.

**THEOREM: (Brody lemma)**
Let $M$ be a compact complex manifold which is not Kobayashi hyperbolic. **Then $M$ contains a Brody curve.**

## Currents and generalized functions

**DEFINITION:** Let $F$ be a Hermitian bundle with connection $\nabla$, on a Riemannian manifold $M$ with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M}\left(|f| + |\nabla f| + ... + |\nabla^k f|\right)$$

the corresponding **$C^k$-norm** defined on smooth sections with compact support. **The $C^k$-topology is independent from the choice of connection and metrics.**

**DEFINITION: A generalized function** is a functional on top forms with compact support, which is continuous in one of $C^i$-topologies.

**DEFINITION: A $k$-current** is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of $C^i$-topologies.

**REMARK:** Currents are forms with coefficients in generalized functions.

## Currents on complex manifolds

**DEFINITION:** The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support).

**CLAIM:** De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

**DEFINITION:** On an complex manifold, **$(p,q)$-currents** are $(p,q)$-forms with coefficients in generalized functions

**REMARK:** **In the literature, this is sometimes called $(n-p,n-q)$-currents**.

**CLAIM:** The Poincare and Poincare Dolbeault-Grothendieck lemma hold on $(p,q)$-currents, and **the $d$- and $\overline{\partial}$-cohomology are the same as for forms.**

## Positive currents

**REMARK:** Positive generalized functions are all $C^0$-continuous as functionals on $C^\infty M$. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and all measures are obtained this way.

**DEFINITION:** Let $\dim_{\mathbb{C}} M = n$. **The cone of positive** $(n-1, n-1)$**-currents** is generated by $\alpha(-\sqrt{-1}\ )^{n-1}\alpha \prod_{i=1}^{n-1} dz_i \wedge d\bar{z}_i$, where $\alpha$ is a non-negative generalized function (that is, a measure), and $z_i$ holomorphic functions.

**REMARK:** An $(n-1, n-1)$-current $\alpha$ on an $n$-dimensional complex manifold is positive if and only if $\int_M \alpha \wedge \beta \geqslant 0$, where $\beta = (-\sqrt{-1}\ )^1 \alpha dz \wedge d\bar{z}$, $z$ a holomorphic function, and $\alpha$ a smooth non-negative function with compact support.

**EXAMPLE: A current of integration $\beta \longrightarrow \int_Z \beta$ is positive,** for any 1-dimensional complex subvariety $Z \subset M$.

**REMARK:** If $Z$ is without boundary, the current of integration $C_Z$ is closed by Stokes' theorem. **If $Z$ has boundary, we have**

$$\langle dC_z, \beta \rangle = \int_{\mathbb{Z}} d\beta = \int_{\partial Z} \beta,$$

**and this is usually non-zero.**

## Ahlfors currents

**THEOREM:** Let $\varphi: \mathbb{C} \longrightarrow M$ be a Brody curve on a complex Hermitian manifold, and $\Delta_r \subset M$ the corresponding disk embeddings. Denote by $A(r)$ the area of $\Delta_r$ in $M$, and let $C_{\Delta_r}$ be its current of integration. **Then there exists a sequence $r_i$ such that $\lim_i A(r_i)^{-1} C_{\Delta_{r_i}}$ converges to a closed current.**

**REMARK:** Any of such limits is called **Ahlfors current**. It is positive, closed, non-zero $(n-1, n-1)$ current, which can be understood as "the current of integration" along the Brody curve.

**Proof. Step 1:** Let $l(r)$ be the length of $\partial\Delta_r$. Using

$$\langle dC_z, \beta\rangle = \int_{\mathbb{Z}} d\beta = \int_{\partial Z} \beta,$$

we obtain that **it suffices to show that $\lim_i \frac{l(r_i)}{A(r_i)} = 0$ for an appropriate sequence $r_i$.**

## Ahlfors currents (2)

**THEOREM:** Let $\varphi : \mathbb{C} \longrightarrow M$ be an entire curve on a complex Hermitian manifold, and $\Delta_r \subset M$ the corresponding disk embeddings. Denote by $A(r)$ the area of $\Delta_r$ in $M$, and let $C_{\Delta_r}$ be its current of integration. **Then there exists a sequence $r_i$ such that $\lim_i A(r)^{-1} C_{\Delta_r}$ converges to a closed current.**

**Step 1:** Let $l(r)$ be the length of $\partial\Delta_r$. Then **it suffices to show that $\lim_i \frac{l(r_i)}{A(r_i)} = 0$ for an appropriate sequence $r_i$.**

**Step 2:** Consider the function $f(x) = |d\varphi|(x)$ on $\mathbb{C}$. Then $A(r) = \int_{\Delta_r} f^2$ and $l(r) = \int_{\partial\Delta_r} f$ (from now on, all integrals are taken with respect to the usual area and length Lebesgue measure on $\mathbb{C}$ and $\partial\Delta_r$). **If such $\{r_i\}$ does not exist, we obtain that $l(r)/A(r) > C$ for some constant $C > 0$.**

**Step 3:** Since $\varphi$ is conformal, the volume of a thin strip $\Delta_r \backslash \Delta_{r-\varepsilon} \subset M$ is approximately equal to $\varepsilon \int_{\partial\Delta_r} f^2$. This gives $\int_{\partial D_r} f^2 = A'(r)$.

**Step 4:** Now we can forget about $M$ entirely. **We are given a positive, bounded function $f$ on $\mathbb{C}$ which satisfies $\int_{\partial D_r} f^2 = A'(r)$, $\int_{\partial D_r} f = l(r)$, and $l(r)/A(r) > C$.** We need to show that this is impossible.

## Ahlfors currents (3)

**Step 4:** Now we can forget about $M$ entirely. **We are given a positive, bounded function $f$ on $\mathbb{C}$ which satisfies $\int_{\partial D_r} f^2 = A'(r)$, $\int_{\partial D_r} f = l(r)$, and $l(r)/A(r) > C$.** We need to show that this is impossible.

**Step 5:** Using Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\left(\int_{\partial D_r} f\right)^2 = l(r)^2 \leqslant 2\pi r \int_{\partial D_r} f^2 = 2\pi r A'(r).$$

Then $l(r) \geqslant CA(r)$ gives $C^2A^2(r) \leqslant 2\pi r A'(r)$. Writing $C_1 = C^2(2\pi)^{-1}$, we obtain $rA'(r) \geqslant A(r)^2 C_1$.

**Step 6:** We have

$$\left(\frac{1}{-A(r)}\right)' = \frac{A'(r)}{A^2(r)} \geqslant \frac{C_1}{r}$$

Integrating both sides, we get

$$-\frac{1}{A(r)} \geqslant C_1 \log(r) - C_2$$

which is impossible, because $A(r)$ is monotonous. ■

## Nevanlinna theory

Let $U$ be a family of closed divisors on a compact complex manifold $M$ projecting surjectively to an open subset of $M$, and $C$ an entire curve. Fix a Hermitian metric on $M$. By Thom transversality theorem, for almost all divisors $D \in U$, $D$ meets $C$ transversally. Let $N_r(D)$ be the number of intersection points of $D$ and the risk $C_r \subset C$. Let $A(C_r)$ be the area of the disk.

**THEOREM:** For almost all $D \in U$, **the limit** $\lim_r \frac{N_r(D)}{A(C_r)}$ **exists and is independent from the choice of $D$.**

**Proof:** Let $A_C$ be the Ahlfors current of $C$. The integration current $[D]$ is not smooth, but its average $[D]_\varphi$ with some positive compactly supported function on $U$ is closed and smooth. Therefore, it can be paired with $A_C$ and gives the intersection number in the cohomology. Taking a sequence of $\varphi$ converging to a $\delta$-function and applying the Fubini theorem, we obtain that the limit $\int [D] \wedge A_C = \lim_r \frac{N_r(D)}{A(C_r)}$ exists and depends on on the class of $D$ in cohomology. ■