Automorphism of algebraically hyperbolic manifolds

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Kobayashi pseudometric

DEFINITION: Pseudometric on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y) = d(y,x)$ and satisfies the triangle inequality $d(x,y) + d(y,z) \geqslant d(x,z)$.

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M: **the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.**

EXAMPLE: The Kobayashi pseudometric on $\mathbb{C}$ vanishes.

CLAIM: Any holomorphic map $X \xrightarrow{\varphi} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.

Proof: If $x \in X$ is connected to x' by a sequence of Poincare disks $\Delta_1, ..., \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), ..., \varphi(\Delta_n)$. ■

CLAIM: (Schwarz-Pick lemma) Any holomorphic map from a disk Δ to itself **is distance-decreasing with respect to the Poincaré metric.**

COROLLARY: The Kobayashi pseudometric on a disk is equal to the Poincaré metric.

Kobayashi hyperbolic manifolds

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since x_i, y_i belongs to Δ, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then $d_K(x, y) \geqslant \max_i d_P(x_i, y_i)$.**

Proof: Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipschitz. ■

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: Any bounded domain Ω in $\mathbb{C}^n$ is Kobayashi hyperbolic.

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is $\geqslant$ that in B. However, the Kobayashi distance in B is bounded by the metric $d(x, y) := \max_i d_P(x_i, y_i)$ as follows from above. ■

Brody theorem

THEOREM: (Brody) A compact complex manifold M **is Kobayashi hyperbolic if and only if any holomorphic map $\mathbb{C} \longrightarrow M$ is constant.**

REMARK: Clearly, **any covering gives a local isometry with respect to the Kobayashi metric.**

COROLLARY: Any non-ramified quotient of a bounded domain **is Kobayashi hyperbolic.**

REMARK: Other examples of hyperbolic manifolds: very general complete intersection subvarieties of very big degree; manifolds of negative holomorphic curvature; period domains for deformations of Hodge structures; Teichmüller space of curves.

REMARK: Examples of non-hyperbolic manifolds: All hyperkähler manifolds are non-hyperbolic. All manifolds containing a rational or elliptic curve are non-hyperbolic. Conjecturally, all Calabi-Yau manifolds are non-hyperbolic.

Algebraically hyperbolic manifolds

DEFINITION: Let M be a projective manifold. We say that M is **algebraically hyperbolic** if there exists $A > 0$ such that for any curve $C \subset M$ of genus g one has $\deg C < A(g-1)$.

REMARK: Algebraically hyperbolic manifolds **contain no elliptic nor rational curves.**

THEOREM: **Kobayashi hyperbolic implies algebraically hyperbolic.**

Converse implication **("algebraically hyperbolic implies Kobayashi hyperbolic")** was **conjectured by J.-P. Demailly** who introduced the notion of algebraic hyperbolicity.

Kobayashi hyperbolicity implies algebraic hyperbolicity

THEOREM: Kobayashi hyperbolic implies algebraically hyperbolic.

Proof. Step 1: Kobayashi metric on a compact curve is a Riemannian metric of constant negative curvature, because the covering map from the disk is an isometry. **Gauss-Bonnet formula implies that its volume is $(2g-2)\alpha$,** where α is some constant independent from the genus.

Step 2: Suppose that M is a projective manifold which is algebraically hyperbolic. Denote by g_{FS} the Fubini-Study metric on M and by g_K the Kobayashi metric on M. By compactness, **there exists a constant ε such that $g_{FS} \leqslant \varepsilon g_K$.**

Step 2: Let $j: S \hookrightarrow M$ be a curve; its genus g is $\geqslant 2$ by hyperbolicity. Denote by g_S the Kobayashi metric on S. Since j is 1-Lipschitz with respect to the Kobayashi metric, one has

$$(2g-2)\alpha = \operatorname{Vol}_{g_S}(S) \geqslant \operatorname{Vol}_{g_K}(S) \geqslant \varepsilon \operatorname{Vol}_{g_{FS}}(S) = \varepsilon \deg S.$$

This gives $g - 1 \geqslant \frac{\varepsilon}{2\alpha} \deg S$, proving hyperbolicity. ■

Automorphism group of an algebraically hyperbolic manifold is discrete

CLAIM: The group of automorphisms of an algebraically hyperbolic manifold M **is discrete.**

Proof. Step 1: The group of automorphisms of a projective manifold is a complex Lie group. If its connected component G_0 is non-trivial, this gives a holomorphic map $\varphi: G_0 \longrightarrow M$. Then G_0 is an extension of an affine group and an abelian variety.

Step 2: Affine algebraic groups are rational varieties, hence the closure of an orbit of an affine algebraic group is unirational and covered by rational curves. An abelian variety is not algebraically hyperbolic because abelian variety of dimension n admits a self-isogeny of order m^n mapping a curve C of genus g to a curve of genus g and degree $m^n \deg C$; the same argument shows that any positive-dimensional orbit of a compact complex commutative Lie group is not algebraically hyperbolic. **Therefore M cannot be algebraically hyperbolic.** ■

Automorphism group of a Kobayashi hyperbolic manifold is finite

An even stronger statement is true for Kobayashi hyperbolic manifolds.

CLAIM: The group of automorphisms of a compact Kobayashi hyperbolic manifold M **is finite.**

Proof: Clearly, the group G of automorphisms of M is closed in its group of isometries (under the Kobayashi metric). The group of isometries of a compact metric space is compact, hence G has only finitely many connected components. Finally, $\dim G_0 = 0$ as shown above. ■

The main result of today's talk

THEOREM: **(joint work with F. Bogomolov and L. Kamenova)**
The group of automorphisms of an algebraically hyperbolic manifold M **is finite.**

Automorphisms of algebraically hyperbolic manifolds: the plan

Theorem 1: The group $\operatorname{Aut}(M)$ of automorphisms of an algebraically hyperbolic manifold M **is finite.**

Plan of the proof.

PROPOSITION: Suppose that the image of $\operatorname{Aut}(M)$ in $GL(H^{1,1}(M,\mathbb{R}))$ does not preserve any rational Kähler class. **Then M is not algebraically hyperbolic.**

REMARK: In this case **the image of $\operatorname{Aut}(M)$ in $GL(H^{1,1}(M,\mathbb{R}))$ is infinite;** indeed, otherwise we take an orbit of a Kähler class and its geometric center is an $\operatorname{Aut}(M)$-invariant Kähler class, because the convex hull of a set of Kähler classes lies in the Kähler cone.

PROPOSITION: Suppose that the image of $\operatorname{Aut}(M)$ in $GL(H^{1,1}(M,\mathbb{R}))$ is finite, but its image in $\operatorname{Aut}(\operatorname{Pic}^0(M))$ is infinite. **Then M is not algebraically hyperbolic.**

PROPOSITION: Suppose that $\operatorname{Aut}(M)$ is infinite, but the image of $\operatorname{Aut}(M)$ in $\operatorname{Aut}(\operatorname{Pic}(M))$ is finite. **Then M is not algebraically hyperbolic.**

Automorphisms acting non-trivially on $H^2(M)$

PROPOSITION: Suppose that the image of Aut(M) in $GL(H^{1,1}(M,\mathbb{R}))$ does not preserve a rational Kähler class. **Then M is not algebraically hyperbolic.**

Proof. Step 1: Let ω be a rational Kähler class which has an infinite Aut(M)-orbit. Replacing ω by $N\omega$, we may assume that ω is a class of a hyperplane section. Then ω^{n-1}, $n = \dim_{\mathbb{C}} M$ is a fundamental class of a smooth complex curve $C \subset M$. Let $f_i(\omega)$ be an orbit of ω, which is infinite by our assumtions. Then

$$\deg_\omega f_i(C) = \int_M \omega \wedge (f_i(\omega))^{n-1} = \int_M f_i^{-1}(\omega) \wedge \omega^{n-1}$$

Since the genus of $f_i(C)$ is constant, from algebraic hyperbolicity **we obtain $\int_M f_i(\omega) \wedge \omega^{n-1} < A$ for some constant $A > 0$.**

Step 2: Let $|\cdot|$ denote the positive definite Hodge-Riemann metric on $H^{1,1}(M)$. Let R be a limit point of the sequence $\frac{f_i(\omega)}{|f_i(\omega)|}$ in the sphere $S \subset H^{1,1}(M)$. Since the sequence $f_i(\omega)$ is infinite, distinct and integral, one has $\lim_i |f_i(\omega)| = \infty$. Then $\int_M f_i(\omega) \wedge \omega^{n-1} < A$ implies that $\int_M R \wedge \omega^{n-1} = 0$. By Hodge-Riemann relations, this gives $\int_M R \wedge R \wedge \omega^{n-2} = -|R|^2 = -1$, hence $\frac{\int_M f_i(\omega) \wedge f_i(\omega) \wedge \omega^{n-2}}{|f_i(\omega)|^2} < -(1-\varepsilon)$ for i sufficiently small. This is a contradiction, because $f_i(\omega)$ is Kähler. ■

Automorphisms acting trivially on Pic(M)

PROPOSITION: Suppose that Aut(M) is infinite, but the image of Aut(M) in Aut(Pic(M)) is finite. **Then M is not algebraically hyperbolic.**

Proof. Step 1: We obtain that an infinite subgroup $\Gamma \subset$ Aut(M) acts trivially on Pic(M). Then it fixes a very ample line bundle $L \in$ Pic(M). We obtain that Γ **acts on $\mathbb{P}H^0(M, L)^*$ preserving the image of the projective embedding $M \longrightarrow \mathbb{P}H^0(M, L)^*$.**

Step 2: Let G be the Zariski closure of Γ in $PGL(H^0(M, L)^*)$. Since Γ acts on M with infinite orbits, the orbits of G are positive-dimensional. This is impossible, because Aut(M) is discrete as shown above. ■

Automorphisms fixing a Kähler class

To prove Theorem 1 it remains only to prove
PROPOSITION: Suppose that the image of $\operatorname{Aut}(M)$ in $GL(H^{1,1}(M,\mathbb{R}))$ is finite, but its image in $\operatorname{Aut}(\operatorname{Pic}^0(M))$ is infinite. **Then M is not algebraically hyperbolic.**

Proof. Step 1: Consider an orbit of a Kähler class. **Its geometric center gives an $\operatorname{Aut}(M)$-invariant Kähler class ω,** because the convex hull of a set of Kähler classes lies in the Kähler cone.

Step 2: The Albanese manifold $\operatorname{Alb}(M) = H^0(\Omega^1 M)^*/H^1(M,\mathbb{Z})$ admits a natural $\operatorname{Aut}(M)$-invariant flat Kähler metric induced by the Hodge-Riemann form on $H^1(M)$. Since $\operatorname{Aut}(M)$ acts on $\operatorname{Alb}(M)$ by isometries, it contains a finite index subgroup Γ acting on $\operatorname{Alb}(M)$ by parallel transport.

Step 3: Let $\operatorname{Par}(\operatorname{Alb}(M))$ be the group of parallel transports of $\operatorname{Alb}(M)$. Since Γ is infinite, it is dense in its closure $T \subset \operatorname{Par}(\operatorname{Alb} M)$, which is positive-dimensional. Take a smooth fiber $\operatorname{Alb}^{-1}(x)$ over $x \in \operatorname{Alb}(M)$. The general fibers of a real analytic map $\operatorname{Alb}^{-1}(T\cdot x) \xrightarrow{\pi} T\cdot x$ are smooth; since all fibers of π exist in dense families, all fibers of π are smooth. Then π is a locally trivial fibration with isomorphic fibers. Passing to the Zariski closure T_1 of T, we obtain an isotrivial fibration $\operatorname{Alb}^{-1}(X) \xrightarrow{\pi_1} X$, where X is an orbit of T_1.

Automorphisms fixing a Kähler class (2)

PROPOSITION: Suppose that the image of $\text{Aut}(M)$ in $GL(H^{1,1}(M,\mathbb{R}))$ is finite, but its image in $\text{Aut}(\text{Pic}^0(M))$ is infinite. **Then M is not algebraically hyperbolic.**

Proof. Step 1: There exists an $\text{Aut}(M)$-invariant Kähler class on M.

Step 2: The group $\text{Aut}(M)$ acts on $\text{Alb}(M)$ by isometries and contains a finite index subgroup Γ acting on $\text{Alb}(M)$ by parallel transport.

Step 3: There is a complex torus $X \subset \text{Alb}(M)$ such that the Albanese map $\text{Alb}^{-1}(X) \xrightarrow{\pi_1} T_1 \cdot x$ is a smooth, isotrivial complex fibration.

Step 4: Isotrivial fibrations with fiber F are classified by $H^1(T_1, \text{Aut}(F))$. Using induction by dimension, we may assume that $\text{Aut}(F)$ is finite. The first cohomology of a torus with coefficients in a finite group is the same as a G-valued local system. Therefore, it becomes trivial after an appropriate finite covering. Then π_1 becomes a trivial fibration after passing to a finite covering $Y \longrightarrow X$, giving a decomposition $\widetilde{\text{Alb}^{-1}(X)} = F \times Y$. This manifold admits self-isogenies of arbitrary high order, giving curves of constant genus and arbitrary high degree in $\text{Alb}^{-1}(X)$. Therefore, $\text{Alb}^{-1}(X) \subset M$ is not algebraically hyperbolic. ■