GAGA theorem for quasihomogeneous singularities

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Chow theorem

REMARK: Algebraic varieties are set of common zeros for a system of polynomial equations. **Complex varieties** are set of common zeros for a system of complex analytic equations. **GAGA** ("Géométrie algébrique et géométrie analytique") is a miracle which happens when analytic implies algebraic.

THEOREM: (Wei-Liang Chow)
Any complex subvariety of $\mathbb{C}P^n$ is algebraic.

Wei-Liang Chow (left) and Shiing-Shen Chern (right) with unidentified colleagues in Germany (circa 1936).

Algebraic structures on complex varieties

DEFINITION: (Formal definition) An algebraic structure on a complex analytic variety Z is a subsheaf of the sheaf of holomorphic functions which can be realized as a sheaf of regular functions for some biholomorphism between Z and a quasi-projective variety.

Less formal definition: Let M be a complex variety which can be embedded to $\mathbb{C}^n$. **Algebraic structure** on M is a finitely generated ring of holomorphic functions on M such that its generators $z_1, ..., z_n$ induce an embedding $M \hookrightarrow \mathbb{C}^n$, and its image is an algebraic variety. In other words, **we fix a dense, finitely-generated subring in the ring $H^0(\mathcal{O}_M)$ of holomorphic functions**.

REMARK: The algebraic structure on a manifold is not unique.

THEOREM: (Zbigniew Jelonek)
There exists an uncountable set of pairwise non-isomorphic algebraic structures on $\mathbb{C} \times S$, where S is an affine complex curve of genus $\geqslant 1$.

REMARK: A complex variety is called **projective** if it admits a holomorphic embedding to $\mathbb{C}P^n$. Chow theorem states that **any projective variety is equipped with a canonical algebraic structure.**

GAGA-style results

Wikipedia: *"Nowadays the phrase "GAGA-style result" is used for any theorem of comparison, allowing passage between a category of objects from algebraic geometry, and their morphisms, to a well-defined subcategory of analytic geometry objects and holomorphic mappings."*

GEOMÉTRIE ALGÉBRIQUE ET GÉOMÉTRIE ANALYTIQUE

par Jean-Pierre SERRE.

INTRODUCTION

Soit X une variété algébrique projective, définie sur le corps des nombres complexes. L'étude de X peut être entreprise de deux points de vue : le point de vue *algébrique*, dans lequel on s'intéresse aux anneaux locaux des points de X, aux applications rationnelles, ou régulières, de X dans d'autres variétés, et le point de vue *analytique* (parfois appelé « transcendant ») dans lequel c'est la notion de fonction holomorphe sur X qui joue le principal rôle. On sait que ce second point de vue s'est révélé particulièrement fécond lorsque X est non singulière,

A contraction

DEFINITION: A Stein variety is a complex subvariety in $\mathbb{C}^n$

DEFINITION: Let M be a topological space, $x \in M$ a marked point. **A contraction** of (M, x) is a continuous map $\varphi: M \to M$ such that for any compact $K \subset M$ and any open $U \ni x$, a sufficiently high iteration of φ satisfies $\varphi^N(K) \subset U$.

Today we will be discussing **invertible holomorphic contractions on Stein varieties**.

EXAMPLE: A linear operator $A: \mathbb{C}^n \to \mathbb{C}^n$ **is a contraction if and only if all its eigenvalues α_i satisfy $|\alpha_i| < 1$.**

EXAMPLE: Let $X \subset \mathbb{C}^n$ be a complex subvariety, preserved by a linear contraction A. **Then A acts on X as a contraction.**

REMARK: An isolated singularity $x \in X$ in a Stein variety is called **quasi-homogeneous** if a neighbourhood of x admits a contraction γ to x. Sometimes one also assumes that the action of γ can be extended to $\mathbb{C}^*$-action. We show that **this condition is automatic.**

Cone varieties

DEFINITION: A **linear Hopf manifold** is a complex manifold $H := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$, where $A \in GL(n, \mathbb{C})$ is an invertible linear contraction.

DEFINITION: A complex variety (X, x, φ) equipped with an invertible holomorphic contraction φ to x is called **a closed cone variety**, and $(X\backslash x, \varphi)$ is **an open cone variety.**

THEOREM: A closed cone variety is always Stein.

REMARK: In dimension > 2, **there is a way to reconstruct a closed cone variety from an open cone,** involving the normalization and Rossi-Andreotti-Siu theorem.

REMARK: Let $(X\backslash x, \varphi)$ be an open cone variety. **Then φ acts on $X\backslash x$ properly discontinuously, and the quotient is a complex variety.**

EXAMPLE: The $\mathbb{Z}$-covering of a complex subvariety in a linear Hopf manifold **is an open cone variety.**

THEOREM: Let $X \subset H$ be a subvariety in a Hopf manifold, and $\tilde{X} \subset \mathbb{C}^n \backslash 0$ its $\mathbb{Z}$-covering. **Then $\tilde{X}$ is an open cone variety**, with the contraction induced by the $\mathbb{Z}$-action on $\mathbb{C}^n \backslash 0$. Moreover, **any open cone variety can be obtained this way.**

Algebraization of holomorphic contractions

THEOREM: Let X be a Stein variety, equipped with an invertible contraction $\varphi : X \to X$. **Then X admits an algebraic structure such that φ is algebraic**. Moreover, **this algebraic structure is unique,** provided that φ is contained in a finite-dimensional connected Lie group acting on X.

THEOREM: In these assumptions, there exists a projective variety $P \subset \mathbb{C}P^n$ such that its preimage under the natural projection $\mathbb{C}^{n+1} \backslash 0 \to \mathbb{C}P^n$ **is biholomorphic to X,** with the contraction φ **induced from the linear contraction on $\mathbb{C}^{n+1}$.**

REMARK: In algebraic geometry, the preimage of P inder the projection $\mathbb{C}^{n+1} \backslash 0 \to \mathbb{C}P^n$ is called **the cone of P**, and its closure **the affine cone**. By Chow theorem, **the affine cone is an algebraic subvariety in $\mathbb{C}^{n+1}$.**

LCK manifolds

The motivation for this research **comes from differential geometry.**

DEFINITION: A complex manifold (M, I) is called **locally conformally Kähler** (LCK) if it admits a covering $(\tilde{M}, I)$ equipped with a Kähler metric $\tilde{\omega}$ such that the deck group of the cover acts on $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties. An **LCK metric** on an LCK manifold is an Hermitian metric on (M, I) such that its pullback to $\tilde{M}$ is conformal with $\tilde{\omega}$.

EXAMPLE: (Ornea-Gauduchon, Ornea-V.) Hopf manifolds are LCK.

REMARK: Equivalently, a Hermitian manifold (M, I, g) is LCK if there exists a closed 1-form θ (called **the Lee form**) such that the fundamental form $\omega(x, y) := g(Ix, y)$ satisfies $d\omega = \theta \wedge \omega$.

REMARK: If $\theta = df$, then $d(e^{-f}\omega) = e^{-f}\omega \wedge \theta - e^{-f}\omega \wedge df = 0$. Therefore, **the form $e^{-f}\omega$ is Kähler.**

THEOREM: (Vaisman) A compact LCK manifold (M, ω, θ) **admits a Kähler metric if and only if θ is exact.**

EXAMPLE: A complex submanifold in an LCK manifold is LCK.

LCK manifolds with potential

DEFINITION: Kähler potential for a Kähler form ω is a function ψ such that $dd^c\psi = \omega$. Here $d^c = IdI^{-1}$. **On a Stein manifold, every Kähler form admits a Kähler potential.**

DEFINITION: An LCK manifold has **a proper LCK potential** if it admits a Kähler $\mathbb{Z}$-covering on which the Kähler metric has a global potential ψ such that the deck group multiplies ψ by a constant (such a function is called **automorphic**). In this case, M is called **an LCK manifold with potential**.

EXAMPLE: All Hopf manifolds are LCK with potential (Gauduchon-Ornea, Ornea-V., 1999-2022). For a classical Hopf manifold $H := (\mathbb{C}^n \setminus 0)/\langle A\rangle$, $A = \lambda \operatorname{Id}$, $|\lambda| > 1$, the flat Kähler metric $\tilde g_0 = \sum dz_i \otimes d\bar z_i$ on $\mathbb{C}^n$ is multiplied by λ^2 by the deck group $\mathbb{Z}$. Also, $\tilde g_0$ has the global automorphic potential $\psi := \sum |z_i|^2$.

EXAMPLE: A complex submanifold in an LCK manifold with potential is LCK with potential.

LCK manifolds with potential are submanifolds in Hopf manifolds

THEOREM: (Ornea-V., 2005)
Let M be an LCK manifold with potential, $\tilde{M}$ its $\mathbb{Z}$-cover, and ψ its LCK potential. **Then ψ is an exhausting strictly plurisubharmonic function on M.** By Andreotti-Rossi, when $\dim_{\mathbb{C}} M \geqslant 3$, **the open cone variety $\tilde{M}$ is associated with a closed cone variety $\tilde{M}_c$, which is equipped with a holomorphic contraction.**

THEOREM: (Ornea-V.)
Let (M, I, ω) be a compact LCK manifold with potential, $\dim_{\mathbb{C}} M \geqslant 3$. **Then (M, I) admits a holomorphic embedding to a linear Hopf manifold.** Conversely, **any submanifold in a linear Hopf manifold is LCK with potential.**

REMARK: Today **I present an algebraic geometric version of these results, valid for singular varieties.**

γ^*-finite functions

DEFINITION: Let $F \in \operatorname{End}(V)$ be an endomorphism of a vector space. A vector $v \in V$ is called **F-finite** if the space generated by $v, F(v), F(F(v)), ...$ is finite-dimensional.

THEOREM: Let $\gamma : X \to X$ be a holomorphic contraction on a Stein variety. **Then $\gamma^* : H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X)$ is a compact operator in topology of uniform convergence on compacts,** Moreover, **the set of γ^*-finite vectors is dense in $H^0(\mathcal{O}_X)$.**

Proof: Compactness of γ^* follows from Montel theorem. Density of γ^*-finite vectors follows from Riesz-Schauder theorem which gives a Jordan cell decomposition for a compact operator on a topological vector space. ■

Algebraic structure on closed cone varieties

PROPOSITION: Let γ be an invertible linear contraction of $\mathbb{C}^n$. **A holomorphic function on $\mathbb{C}^n$ is γ-finite if and only if it is polynomial.**

Proof: Clearly, a polynomial function is γ-finite. The operator γ^* acts on homogeneous polynomials of degree d with eigenvalues $\alpha_{i_1}\alpha_{i_2}...\alpha_{i_d}$, where α_{i_j} are the eigenvalues of γ on $\mathbb{C}^n$. Since γ is a contraction, all α_{i_j} satisfy $|\alpha_{i_j}| < 1$. Therefore, any sequence $\{\alpha_{i_1}\alpha_{i_2}...\alpha_{i_d}\}$ converges to 0 as d goes to infinity. We obtain that every given number can be realized as an eigenvalue of γ^* on homogeneous polynomials of degree d for at most finitely many choices of d. Therefore, any root vector of γ^* is a finite sum of homogeneous polynomials. ■

REMARK: This theorem gives algebraic structure on closed cone variety: **the γ^*-finite functions are regular functions in this algebraic structure.**

Cone varieties to Hopf manifolds

THEOREM: Let $(\tilde{M}_c, \gamma, c)$ be a cone variety, and $\tilde{M}$ the corresponding open cone variety. **Then there exists a holomorphic embedding $j: \tilde{M}/\langle\gamma\rangle \hookrightarrow H$ to a Hopf manifold.**

Proof. Step 1: Let R be the ring of γ-finite holomorphic functions on $\tilde{M}_c$, I the maximal ideal of c, and and $V \subset I$ a finite-dimensional γ-invariant space generating I. As shown above, the action of γ^* is compact on I and has all eigenvalues < 1. Then R is dense in $\mathcal{O}_{\tilde{M}_c}$. **Therefore, the functions in V separate the points in $\tilde{M}$, for V sufficiently big.**

Step 2: By Cauchy formula, R is dense in C^1-topology whenever it is dense in C^0-topology; therefore, the differentials of the functions $f \in R$ generate $T_x^*\tilde{M}$ for all $x \in \tilde{M}$. **This implies that the tautological map $\tilde{M} \to V^*$, taking $x \in \tilde{M}$, $v \in V$ to $v(x)$ is a holomorphic embedding.**

Step 3: This map is by construction γ-equivariant, hence it induces a holomorphic embedding $\tilde{M}/\langle\gamma\rangle \hookrightarrow H = V^*/\langle\gamma\rangle$. ■

GAGA theorem and filtrability

REMARK: "Coherent sheaves" is essentially vector bundles allowed to have singularities. In algebraic category, a vector bundle is given by algebraic transition functions; in complex analytic category, it is given by complex analytic transition functions. Serre proved that **over a projective variety these notions are equivalent.**

THEOREM: **(Serre)** Let $\mathcal{F}$ be a complex analytic coherent sheaf over a projective variety X. **Then $\mathcal{F}$ admits a natural algebraic structure.** Moreover, the category of complex analytic coherent sheaves on X **is equivalent to the category of algebraic coherent sheaves on X.**

DEFINITION: A coherent sheaf is **filtrable** if it admits a filtration with rank 1 quotient sheaves.

REMARK: **Algebraicity easily implies filtrability.** Essentially, filtrable coherent sheaves are extensions of line bundles, and we understand the category of filtrable coherent sheaves quite well. However, filtrability is a very strong property! **It fails on all non-algebraic surfaces.**

REMARK: The aim of this work: **using a version of GAGA theorem, prove filtrability of coherent sheaves for complex varieties which admit an embedding to a Hopf manifold.**

GAGA theorem for γ-equivariant coherent sheaves

REMARK: A coherent sheaf is called **reflexive** if it is isomorphic to its double dual; a dual of a coherent sheaf is always reflexive. Note that $\langle\gamma\rangle$-equivariant analytic reflexive coherent sheaf $\mathcal{F}$ on X are the same as reflexive coherent sheaf $\mathcal{F}$ on $M := X_0/\langle\gamma\rangle$.

THEOREM: (GAGA theorem for varieties with contractions) Let (X, γ) be a closed cone variety. Using the algebraization theorem, we consider X as an algebraic variety. Consider a $\langle\gamma\rangle$-equivariant analytic reflexive coherent sheaf $\mathcal{F}$ on X. **Then $\mathcal{F}$ admits an algebraic structure, which is uniquely determined by the γ-equivariant structure.**

THEOREM: (filtrability of coherent sheaves) Let (X_0, γ) be a cone variety and $\mathcal{F}$ a γ-equivariant coherent reflexive sheaf on X_0, or, equivalently, a coherent sheaf on $X_0/\langle\gamma\rangle$. Assume that $\dim_{\mathbb{C}} X_0 \geqslant 3$. **Then $\mathcal{F}$ admits a γ-equivariant filtration with all associated graded subquotients γ-equivariant coherent sheaves of rank $\leqslant 1$.**

REMARK: Every cone variety is a cone over a projective variety P. We prove this theorem by reducing the coherent sheaves on $M := X_0/\langle\gamma\rangle$ to coherent sheaves on this projective variety. We describe how to reconstruct any coherent sheaf on M from coherent sheaves on P. However, **so far we failed to produce a functorial correspondence,** though there must be one.