Algebraic cones as complex varieties

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Algebraic structures on complex varieties

DEFINITION: (Formal definition) An algebraic structure on a complex analytic variety Z is a subsheaf of the sheaf of holomorphic functions which can be realized as a sheaf of regular functions for some biholomorphism between Z and a quasi-projective variety.

Less formal definition: Let M be a complex variety which can be embedded to $\mathbb{C}^n$. **Algebraic structure** on M is a finitely generated ring of holomorphic functions on M such that its generators $z_1, ..., z_n$ induce an embedding $M \hookrightarrow \mathbb{C}^n$, and its image is an algebraic variety. In other words, **we fix a dense, finitely-generated subring in the ring $H^0(\mathcal{O}_M)$ of holomorphic functions**.

REMARK: The algebraic structure on a manifold is not unique.

EXAMPLE: (C. Simpson)
The manifold $\mathbb{C}^* \times \mathbb{C}^*$ **admits an algebraic structure without non-constant global regular functions.**

THEOREM: (Zbigniew Jelonek)
There exists an uncountable set of pairwise non-isomorphic algebraic structures on $\mathbb{C} \times S$, where S is an affine complex curve of genus $\geqslant 1$.

A contraction

DEFINITION: A Stein variety is a complex subvariety in $\mathbb{C}^n$

REMARK: We shall always tacitly assume that our Stein varieties have isolated singularities.

DEFINITION: Let M be a topological space, $x \in M$ a marked point. **A contraction** of (M,x) is a continuous map $\varphi: M \to M$ such that for any compact $K \subset M$ and any open $U \ni x$, a sufficiently high iteration of φ satisfies $\varphi^N(K) \subset U$.

EXAMPLE: A linear operator $A: \mathbb{C}^n \to \mathbb{C}^n$ **is a contraction if and only if all its eigenvalues α_i satisfy $|\alpha_i| < 1$.**

EXAMPLE: Let $X \subset \mathbb{C}^n$ be a complex subvariety, preserved by a linear contraction A. **Then A acts on X as a contraction.**

The main results of this talk:

THEOREM: Let X be a Stein variety with an most one singular point equipped with an invertible contraction $\varphi: X \to X$. **Then X admits an algebraic structure such that φ is algebraic.** Moreover, **this algebraic structure is unique.**

THEOREM: In these assumptions, **there exists a projective orbifold P an an ample bundle L such that X is isomorphic to the spectrum of the ring $\bigoplus_{i=0}^\infty H^0(X, L^i)$.** Moreover, P can be chosen in such a way that **the action of φ on $\bigoplus_{i=0}^\infty H^0(X, L^i)$ is obtained from an automorphism of X which acts on L equivariantly.**

REMARK: Let $x \in X$ be the fixed point of the contraction φ, and $X_0 := X \backslash x$. **The isomorphism $X = \operatorname{Spec}(\bigoplus_{i=0}^\infty H^0(X, L^i))$ is equivalent to X_0 being isomorphic to the space of all non-zero vectors in the total space $\operatorname{Tot}(L)$.**

REMARK: However, **the pair (P, L) is not determined by X and its algebraic structure uniquely:** the same X might be obtained from different projective orbifolds. Example: $X = \mathbb{C}^n$, $\varphi(x) := \frac{1}{2}x$, and and $P = \frac{\mathbb{C}^n \backslash 0}{\mathbb{C}^*}$ any of the weighted projective spaces.

Stein completion

THEOREM: (a version of Hartogs theorem)
Let X be a normal Stein variety, $\dim_{\mathbb{C}} X > 1$, and $K \subset X$ a compact subset. **Then every holomorphic function on $X \backslash K$ can be extended to X.**

DEFINITION: Let A be a commutative Fréchet algebra over $\mathbb{C}$. The **continuous spectrum** Spec(A) of A is defined as the set of all continuous $\mathbb{C}$-linear homomorphisms $A \to \mathbb{C}$.

THEOREM: (O. Forster, 1966, 1967)
Let X be a Stein variety, and $H^0(\mathcal{O}_X)$ is the algebra of holomorphic functions equipped with the topology of uniform convergence on compacts. **Then** $\operatorname{Spec}(H^0(\mathcal{O}_X)) = X$.

DEFINITION: Let X be a normal Stein variety, and $K \subset X$ a compact subset. By Hartogs, the ring of functions on X is identified with $H^0(\mathcal{O}_{X \backslash K})$; by Forster, this ring with its C^0 topology uniquely defines X. Following Andreotti-Siu, we call X **the Stein completion** of $X \backslash K$. If $X \backslash K$ is smooth, ts Stein completion **is a normal Stein variety with isolated singularities**.

Algebraic cones

DEFINITION: Let P be a projective orbifold, and L an ample line bundle on P. Assume that the total space $\operatorname{Tot}^\circ(L)$ of all non-zero vectors in L is smooth. **An open algebraic cone** is $\operatorname{Tot}^\circ(L)$.

DEFINITION: The corresponding **closed algebraic cone** is its Stein completion Z;

EXAMPLE: Let $P \subset \mathbb{C}P^n$, and $L = \mathcal{O}(1)|_P$. Then **the open algebraic cone** $\operatorname{Tot}^\circ(L)$ **can be identified with the set** $\pi^{-1}(P)$ of all $v \in \mathbb{C}^{n+1} \backslash 0$ projected to P under the standard map $\pi: \mathbb{C}^{n+1} \backslash 0 \to \mathbb{C}P^n$. **The closed algebraic cone is the normalization of its closure in** $\mathbb{C}^{n+1}$.

REMARK: The closed algebraic cone **is obtained by adding one point, called "the apex", or "the origin", to** $\operatorname{Tot}^\circ(L)$.

REMARK: The structure of a complex variety on this one-point completion **is unique only in the assumption of normality**. Without normality, **it is not unique**. Normality of the closed cone $C(P)$ **is equivalent to the "projective normality" of** P.

Algebraic cones and subvarieties in Hopf manifolds

DEFINITION: A **linear Hopf manifold** is a complex manifold $H := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$, where $A \in GL(n, \mathbb{C})$ is an invertible linear contraction.

THEOREM: Let $M \subset H$ be a submanifold in a Hopf manifold, and $\tilde{M} \subset \mathbb{C}^n \backslash 0$ its $\mathbb{Z}$-covering. **Then $\tilde{M}$ is an open algebraic cone**, with the contraction induced by the $\mathbb{Z}$-action on $\mathbb{C}^n \backslash 0$. Moreover, **any open algebraic cone can be obtained this way.**

DEFINITION: A complex manifold (M, I) is called **locally conformally Kähler** (LCK) if it admits a covering $(\tilde{M}, I)$ equipped with a Kähler metric $\tilde{\omega}$ such that the deck group of the cover acts on $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties. An **LCK metric** on an LCK manifold is an Hermitian metric on (M, I) such that its pullback to $\tilde{M}$ is conformal with $\tilde{\omega}$.

EXAMPLE: (Ornea-Gauduchon, Ornea-V.) Hopf manifolds are LCK.

LCK manifolds with potential

DEFINITION: **Kähler potential** for a Kähler form ω is a function ψ such that $dd^c\psi = \omega$. Here $d^c = IdI^{-1}$. **On a Stein manifold, every Kähler form admits a Kähler potential.**

DEFINITION: An LCK manifold has **a proper LCK potential** if it admits a Kähler $\mathbb{Z}$-covering on which the Kähler metric has a global potential ψ such that the deck group multiplies ψ by a constant (such a function is called **automorphic**). In this case, M is called **an LCK manifold with potential**.

EXAMPLE: All Hopf manifolds are LCK with potential (Gauduchon-Ornea, Ornea-V., 1999-2022). For a classical Hopf manifold $H := (\mathbb{C}^n \setminus 0)/\langle A\rangle$, $A = \lambda \operatorname{Id}$, $|\lambda| > 1$, the flat Kähler metric $\tilde{g}_0 = \sum dz_i \otimes d\bar{z}_i$ on $\mathbb{C}^n$ is multiplied by λ^2 by the deck group $\mathbb{Z}$. Also, $\tilde{g}_0$ has the global automorphic potential $\psi := \sum |z_i|^2$.

EXAMPLE: A complex submanifold in an LCK manifold with potential is LCK with potential.

LCK manifolds with potential are submanifolds in Hopf manifolds

THEOREM: (Ornea-V., 2005)
Let M be an LCK manifold with potential, $\tilde{M}$ its $\mathbb{Z}$-cover, and ψ its LCK potential. **Then ψ is an exhausting strictly plurisubharmonic function on M.** By Andreotti-Rossi, when $\dim_{\mathbb{C}} M \geqslant 3$, **the manifold $\tilde{M}$ admits a Stein completion $\tilde{M}_c$, which is equipped with a holomorphic contraction.** Moreover, **$\tilde{M}_c$ is obtained from $\tilde{M}$ by adding precisely one point.**

THEOREM: (Ornea-V.)
Let (M, I, ω) be a compact LCK manifold with potential, $\dim_{\mathbb{C}} M \geqslant 3$. **Then (M, I) admits a holomorphic embedding to a linear Hopf manifold.** Conversely, **any submanifold in a linear Hopf manifold is LCK with potential.**

γ^*-finite functions

DEFINITION: Let $F \in \operatorname{End}(V)$ be an endomorphism of a vector space. A vector $v \in V$ is called **F-finite** if the space generated by $v, F(v), F(F(v)), ...$ is finite-dimensional.

THEOREM: Let $\gamma: X \to X$ be a holomorphic contraction on a Stein variety. **Then $\gamma_* : H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X)$ is a compact operator in topology of uniform convergence on compacts,** Moreover, **the set of γ^*-finite vectors is dense in $H^0(\mathcal{O}_X)$.**

LEMMA: Let γ be an invertible linear contraction of $\mathbb{C}^n$. **A holomorphic function on $\mathbb{C}^n$ is γ-finite if and only if it is polynomial.**

Proof: Clearly, a polynomial function is γ-finite. The operator γ^* acts on homogeneous polynomials of degree d with eigenvalues $\alpha_{i_1}\alpha_{i_2}...\alpha_{i_d}$, where α_{i_j} are the eigenvalues of γ on $\mathbb{C}^n$. Since γ is a contraction, all α_{i_j} satisfy $|\alpha_{i_j}| < 1$. Therefore, any sequence $\{\alpha_{i_1}\alpha_{i_2}...\alpha_{i_d}\}$ converges to 0 as d goes to infinity. We obtain that every given number can be realized as an eigenvalue of γ^* on homogeneous polynomials of degree d for finitely many choices of d only. Therefore, any root vector of γ^* is a finite sum of homogeneous polynomials. ■

γ^*-finite functions (2)

COROLLARY: Let $M \hookrightarrow H$ be a complex subvariety of a linear Hopf manifold, and $\tilde{M}_c \to \mathbb{C}^N$ the corresponding map of weak Stein completions, with $\tilde{M}_c$ obtained as the closure of $\tilde{M} \subset \mathbb{C}^N$ by adding the zero. **Then $\tilde{M}_c$ is an algebraic subvariety, that is, a set of common zeroes of a system of polynomial equations.**

Proof: The γ^*-finite functions are dense in the ideal of M_c; therefore, this ideal is generated by polynomials. ■

Weighted projective spaces

Recall that any representation V of $\mathbb{C}^*$ is a direct sum of 1-dimensional representations isomorphic to ρ_w, with $\mathbb{C}^*$ acting by $\rho_w(t)(z) = t^w z$. Such a representation is called **representation of weight w**.

CLAIM: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$. Assume that ρ contains a contraction. **Then all weights of ρ are positive or negative.** ■

CLAIM: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, ..., w_n \in \mathbb{Z}^{>0}$. **Then its orbit space $\mathbb{C}P^{n-1}(w_1, ..., w_n)$ is equipped with a structure of a projective orbifold,** and $\mathbb{C}^n \backslash 0$ **can be identified with the total space of an ample $\mathbb{C}^*$-bundle over $\mathbb{C}P^{n-1}(w_1, ..., w_n)$.** ■

DEFINITION: The orbifold $\mathbb{C}P^{n-1}(w_1, ..., w_n)$ is called **the weighted projective space**.

CLAIM: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, ..., w_n \in \mathbb{Z}^{>0}$, and $Z \subset \mathbb{C}^n \backslash 0$ be a ρ-invariant submanifold. **Then the orbit space $Z/\mathbb{C}^*$ is a projective orbifold in the corresponding weighted projective space $\mathbb{C}P^{n-1}(w_1, ..., w_n)$.**
■

COROLLARY: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, ..., w_n \in \mathbb{Z}^{>0}$, and $Z \subset \mathbb{C}^n \backslash 0$ a ρ-invariant submanifold. **Then Z is an open algebraic cone.** ■

Stein varieties equipped with a contraction are algebraic cones

COROLLARY: Let $\tilde{M}_c$ be a normal Stein variety equipped with a holomorphic contraction γ, with the only singularity at the origin, denoted c. **Then $\tilde{M}_c$ admits a structure of an algebraic cone.**

Proof. Step 1: Let $\tilde{M} := \tilde{M}_c \backslash c$. Then $\frac{\tilde{M}}{\langle\gamma\rangle}$ admits a holomorphic embedding to a linear Hopf manifold $\frac{\mathbb{C}^n \backslash 0}{\langle A\rangle}$. **This implies that the ideal of $\tilde{M}_c$ in $\mathcal{O}_{\mathbb{C}^n}$ is generated by A^*-finite functions, that is, polynomials,** hence $\tilde{M}_c \subset \mathbb{C}^n$ is an affine subvariety.

Step 2: Let $\mathcal{G}_A$ be a connected component of the algebraic closure of $\langle A\rangle$. A connected abelian algebraic group $\mathcal{G}_A \subset GL(n,\mathbb{C})$ is isomorphic to $(\mathbb{C}^*)^k \times \mathbb{C}^l$, where each $\mathbb{C}^*$ acts diagonally, and $\mathbb{C}$ are unipotent subgroups. Therefore, $\tilde{M}_c \subset \mathbb{C}^n$ is $\mathbb{C}^*$-invariant, for some $\mathbb{C}^*$ acting by contractions. **This produces a $\mathbb{C}^*$-fibration $\tilde{M} \to \frac{\tilde{M}}{\mathbb{C}^*}$ with the quotient $\frac{\mathbb{C}^n \backslash 0}{\mathbb{C}^*}$ identified with the weighted projective space.** ■