Algebraic structures on affine cones and non-Kähler geometry

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Algebraic structures on complex varieties

DEFINITION: (Formal definition) An algebraic structure on a complex analytic variety Z is a subsheaf of the sheaf of holomorphic functions which can be realized as a sheaf of regular functions for some biholomorphism between Z and a quasi-projective variety.

Less formal definition: Let M be a complex variety which can be embedded to $\mathbb{C}^n$. **Algebraic structure** on M is a finitely generated ring of holomorphic functions on M such that its generators $z_1, ..., z_n$ induce an embedding $M \hookrightarrow \mathbb{C}^n$, and its image is an algebraic variety. In other words, **we fix a dense, finitely-generated subring in the ring $H^0(\mathcal{O}_M)$ of holomorphic functions**.

REMARK: The algebraic structure on a manifold is not unique.

EXAMPLE: (C. Simpson)
The manifold $\mathbb{C}^* \times \mathbb{C}^*$ **admits an algebraic structure without global regular functions.**

THEOREM: (Zbigniew Jelonek)
There exists an uncountable set of pairwise non-isomorphic algebraic structures on $\mathbb{C} \times S$, where S is a complex curve of genus $\geqslant 1$.

A contraction

DEFINITION: A Stein variety is a complex subvariety in $\mathbb{C}^n$

REMARK: We shall always tacitly assume that our Stein varieties have isolated singularities.

DEFINITION: Let M be a topological space, $x \in M$ a marked point. **A contraction** of (M, x) is a continuous map $\varphi: M \to M$ such that for any compact $K \subset M$ and any open $U \ni x$, a sufficiently high iteration of φ satisfies $\varphi^N(K) \subset U$.

EXAMPLE: A linear operator $A: \mathbb{C}^n \to \mathbb{C}^n$ **is a contraction if and only if all its eigenvalues α_i satisfy $|\alpha_i| < 1$.**

EXAMPLE: Let $X \subset \mathbb{C}^n$ be a complex subvariety, preserved by a linear contraction A. **Then A acts on X as a contraction.**

The main results of this talk:

THEOREM: Let X be a Stein variety with an most one singular point equipped with an invertible contraction $\varphi: X \to X$. **Then X admits an algebraic structure such that φ is algebraic**. Moreover, **this algebraic structure is unique.**

THEOREM: In these assumptions, **there exists a projective orbifold P an an ample bundle L such that X is isomorphic to the spectrum of the ring $\bigoplus_{i=0}^\infty H^0(X, L^i)$.** Moreover, P can be chosen in such a way that **the action of φ on $\bigoplus_{i=0}^\infty H^0(X, L^i)$. is obtained from an automorphism of X which acts on L equivariantly.**

REMARK: Let $x \in X$ be the fixed point of the contraction φ, and $X_0 := X \backslash x$. **The isomorphism $X = \operatorname{Spec}(\bigoplus_{i=0}^\infty H^0(X, L^i))$ is equivalent to X_0 being isomorphic to the space of all non-zero vectors in the total space $\operatorname{Tot}(L)$.**

REMARK: However, **the pair (P, L) is not determined by X and its algebraic structure uniquely:** the same X might be obtained from different projective orbifolds. Example: $X = \mathbb{C}^n$, $\varphi(x) := \frac{1}{2}x$, and and $P = \frac{\mathbb{C}^n \backslash 0}{\mathbb{C}^*}$ any of the weighted projective spaces.

Stein completion

THEOREM: (a version of Hartogs theorem)
Let X be a normal Stein variety, $\dim_{\mathbb{C}} X > 1$, and $K \subset X$ a compact subset. **Then every holomorphic function on $X \backslash K$ can be extended to X.**

DEFINITION: Let A be a commutative Fréchet algebra over $\mathbb{C}$. The **continuous spectrum** $\operatorname{Spec}(A)$ of A is defined as the set of all continuous $\mathbb{C}$-linear homomorphisms $A{\to}\mathbb{C}$.

THEOREM: (O. Forster, 1966, 1967)
Let X be a Stein variety, and $H^0(\mathcal{O}_X)$ is the algebra of holomorphic functions equipped with the topology of uniform convergence on compacts. **Then** $\operatorname{Spec}(H^0(\mathcal{O}_X)) = X$**.**

DEFINITION: Let X be a normal Stein variety, and $K \subset X$ a compact subset. By Hartogs, the ring of functions on X is identified with $H^0(\mathcal{O}_{X\backslash K})$; by Forster, this ring with its C^0 topology uniquely defines X. Following Andreotti-Siu, we call X **the Stein completion** of $X \backslash K$.

Algebraic cones

DEFINITION: Let P be a projective orbifold, and L an ample line bundle on P. Assume that the total space $\operatorname{Tot}^\circ(L)$ of all non-zero vectors in L is smooth. **An open algebraic cone** is $\operatorname{Tot}^\circ(L)$.

DEFINITION: The corresponding **closed algebraic cone** is its Stein completion Z.

EXAMPLE: Let $P \subset \mathbb{C}P^n$, and $L = \mathcal{O}(1)|_P$. Then **the open algebraic cone** $\operatorname{Tot}^\circ(L)$ **can be identified with the set** $\pi^{-1}(P)$ of all $v \in \mathbb{C}^{n+1}\backslash 0$ projected to P under the standard map $\pi: \mathbb{C}^{n+1}\backslash 0 \to \mathbb{C}P^n$. **The closed algebraic cone is the normalization of its closure in** $\mathbb{C}^{n+1}$**.**

REMARK: The closed algebraic cone **is obtained by adding one point, called "the apex", or "the origin", to** $\operatorname{Tot}^\circ(L)$**.**

Algebraic cones and subvarieties in Hopf manifolds

DEFINITION: A **linear Hopf manifold** is a complex manifold $H := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$, where $A \in GL(n, \mathbb{C})$ is an invertible linear contraction. **A classical Hopf manifold** is a linear Hopf manifold such that A is a scalar matrix.

THEOREM: Let $M \subset H$ be a submanifold in a Hopf manifold, and $\tilde{M} \subset \mathbb{C}^n \backslash 0$ its $\mathbb{Z}$-covering. **Then $\tilde{M}$ is an open algebraic cone. Moreover, any open algebraic cone can be obtained this way.**

Proof: Later today

Compact operators

DEFINITION: Recall that a subset X of a topological space Y is called **precompact**, if its closure is compact.

DEFINITION: A subset $K \subset V$ of a topological vector space is called **bounded** if for any open set $U \ni 0$, there exists a number $\lambda_U \in \mathbb{R}^{>0}$ such that $\lambda_U K \subset U$.

DEFINITION: Let V, W be topological vector spaces. A continuous operator $\varphi: V \to W$ is called **compact** if the image of any bounded set is precompact.

THEOREM: Let M be a complex manifold, and let $H^0_b(\mathcal{O}_M)$ the space of all bounded holomorphic functions, equipped with the sup-norm $|f|_{\sup} := \sup_M |f|$. **Then $H^0_b(\mathcal{O}_M)$ is a Banach space.**

Proof: The space of bounded continuous functions is Banach with respect to the sup-norm. However, The uniform convergence of holomorphic functions implies their C^∞-convergence by Cauchy theorem. ■

The Banach space of bounded holomorphic functions

THEOREM: Let X be a complex variety, and $\gamma : X \to X$ a holomorphic contraction to $x \in X$ such that $\gamma(X)$ is precompact. Consider the Banach space $V = H^0_b(\mathcal{O}_X)$ of bounded holomorphic functions with the sup-norm, and let $V_x \subset V$ be the space of all $v \in V$ vanishing in x. **Then $\gamma^* : V \to V$ is compact, and the eigenvalues of its restriction to V_x are strictly less than 1 in absolute value.**

Proof. Step 1: For any $f \in H^0(\mathcal{O}_X)$ we have $|\gamma^* f|_{\sup} = \sup_{x \in \overline{\gamma(X)}} |f(x)|$. **This implies that $\gamma^*(f)$ is bounded.**

Step 2: Consider the space $H^0(\mathcal{O}_X)$ of holomorphic functions with topology of the uniform convergence on compact subsets. By Montel's theorem, the identity map $H^0_b(\mathcal{O}_X) \to H^0(\mathcal{O}_X)$ is compact. By Step 1, this implies that $\gamma^*(f)$ is precompact in sup-topology. **Therefore, γ^* takes bounded sets to precompact.**

Step 3: Let x be the fixed point of γ. All derivatives in x of the sequence $f, \gamma^* f, (\gamma^*)^2 f, (\gamma^*)^3 f, \ldots$ converge to zero; therefore, this sequence converges to a constant. This implies that γ^* acts on V_x as a contraction, hence **all its eigenvalues are < 1.** ■

The Riesz-Schauder theorem

THEOREM: (Riesz-Schauder theorem)
Let $F: V \to V$ be a compact operator on a Banach space. **Then for each non-zero $\mu \in \mathbb{C}$, there exists a sufficiently big number $N \in \mathbb{Z}^{\gg 0}$ such that for each $n > N$ one has $V = \ker(F - \mu \operatorname{Id})^n \oplus \overline{\operatorname{im}(F - \mu \operatorname{Id})^n}$, where $\overline{\operatorname{im}(F - \mu \operatorname{Id})^n}$ is the closure of the image.** Moreover, $\ker(F - \mu \operatorname{Id})^n$ **is finite-dimensional and independent on $n > N$.** ∎

REMARK: Define **the root space of an operator $F \in \operatorname{End}(V)$, associated with an eigenvalue μ,** as $\bigcup_{n \in \mathbb{Z}} \ker(F - \mu \operatorname{Id})^n$. In the finite-dimensional case, the root spaces coincide with the Jordan cells of the corresponding matrix. The Riesz-Schauder theorem can be reformulated by saying that any compact operator $F \in \operatorname{End}(V)$ admits a Jordan cell decomposition, **with V identified with a completed direct sum of the root spaces, which are all finite-dimensional;** moreover, **the eigenvalues of F converge to zero.**

DEFINITION: Let $F \in \operatorname{End}(V)$ be an endomorphism of a vector space. A vector $v \in V$ is called **F-finite** if the space generated by $v, F(v), F(F(v)), ...$ is finite-dimensional.

COROLLARY: Let $F: V \to V$ be a compact operator on a Banach space, and $V_0 \subset V$ the space of all F-finite vectors. **Then V_0 is dense in V.** ∎

γ^*-finite functions on complex manifolds

LEMMA: Let γ be an invertible linear contraction of $\mathbb{C}^n$. **A holomorphic function on $\mathbb{C}^n$ is γ-finite if and only if it is polynomial.**

Proof: Clearly, a polynomial function is γ-finite. The operator γ^* acts on homogeneous polynomials of degree d with eigenvalues $\alpha_{i_1}\alpha_{i_2}...\alpha_{i_d}$, where α_{i_j} are the eigenvalues of γ on $\mathbb{C}^n$. Since γ is a contraction, all α_{i_j} satisfy $|\alpha_{i_j}| < 1$. Therefore, any sequence $\{\alpha_{i_1}\alpha_{i_2}...\alpha_{i_d}\}$ converges to 0 as d goes to infinity. We obtain that every given number can be realized as an eigenvalue of γ^* on homogeneous polynomials of degree d for finitely many choices of d only. Therefore, any root vector of γ^* is a finite sum of homogeneous polynomials. ■

COROLLARY: Let $M \hookrightarrow H$ be a complex subvariety of a linear Hopf manifold, and $\tilde{M}_c \rightarrow \mathbb{C}^N$ the corresponding map of weak Stein completions, with $\tilde{M}_c$ obtained as the closure of $\tilde{M} \subset \mathbb{C}^N$ by adding the zero. **Then $\tilde{M}_c$ is an algebraic subvariety, that is, a set of common zeroes of a system of polynomial equations.**

Proof: Using Riesz-Schauder, we prove that the ideal of $\tilde{M}_c \subset \mathbb{C}^n$ is generated by γ^*-finite functions, which are polynomial, as shown above. ■

Contractions are properly discontinuous

LEMMA: Let γ act on a complex variety $\tilde{M}_c$ by contractions, contracting $\tilde{M}_c$ to a point c. **Then the corresponding $\mathbb{Z}$-action on $\tilde{M} := \tilde{M}_c \backslash \{c\}$ is properly discontinuous, hence $\tilde{M}/\mathbb{Z}$ is Hausdorff; it is a manifold when $\tilde{M}$ is a manifold.**

Proof: By definition, the $\mathbb{Z}$-action is properly discontinuous if every point has a neighbourhood U such that the set $\{g \in \mathbb{Z} \ \ | \ \ g(U) \cap U \neq \emptyset\}$ is finite. Let $x \in \tilde{M}$ and K be the compact closure of an open neighbourhood of $U \subset \tilde{M}$ containing x. Since $\tilde{M}_c$ is Hausdorff, there exists a neighbourhood $W \ni c$ such that its closure does not intersect K. By definition of contractions, there exists $N > 0$ such that $\gamma^n(K) \subset W$ for all $n \geqslant N$. This implies that $\gamma^n(K) \cap K = \emptyset$. This also implies that $K \cap \gamma^{-n}(K) = \emptyset$. **We have shown that $\gamma^n(K) \cap K = \emptyset$ for all $n \notin [-N, N]$.** ∎

Stein varieties to Hopf manifolds

THEOREM: Let $\tilde{M}_c$ be a Stein variety equipped with a holomorphic contraction γ, contracting it to the point c. Assume that the complement $\tilde{M} := \tilde{M}_c \backslash c$ is smooth. **Then there exists a holomorphic embedding $j: \tilde{M}/\langle\gamma\rangle \hookrightarrow H$ to a Hopf manifold.**

Proof. Step 1: Let R be the ring of γ-finite holomorphic functions on $\tilde{M}_c$, I the maximal ideal of c, and and $V \subset I$ a finite-dimensional γ-invariant space generating I. As shown above, the action of γ^* is compact on I and has all eigenvalues < 1. By Riesz-Schauder theorem, R is dense in $\mathcal{O}_{\tilde{M}_c}$. **Therefore, the functions in V separate the points in $\tilde{M}$, for V sufficiently big.**

Step 2: By Cauchy formula, R is dense in C^1-topology whenever it is dense in C^0-topology; therefore, the differentials of the functions $f \in R$ generate $T_x^*\tilde{M}$ for all $x \in \tilde{M}$. **This implies that the tautological map $\tilde{M} \to V^*$, taking $x \in \tilde{M}$, $v \in V$ to $v(x)$ is a holomorphic embedding.**

Step 3: This map is by construction γ-equivariant, hence it induces a holomorphic embedding $\tilde{M}/\langle\gamma\rangle \hookrightarrow H = V^*/\langle\gamma\rangle$. ■

Weighted projective spaces

Recall that any representation V of $\mathbb{C}^*$ is a direct sum of 1-dimensional representations isomorphic to ρ_w, with $\mathbb{C}^*$ acting by $\rho_w(t)(z) = t^w z$. Such a representation is called **representation of weight w**.

CLAIM: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$. Assume that ρ contains a contraction. **Then all weights of ρ are positive or negative.** ■

CLAIM: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, ..., w_n \in \mathbb{Z}^{>0}$. **Then its orbit space $\mathbb{C}P^{n-1}(w_1, ..., w_n)$ is equipped with a structure of a projective orbifold,** and $\mathbb{C}^n \backslash 0$ **can be identified with the total space of an ample $\mathbb{C}^*$-bundle over $\mathbb{C}P^{n-1}(w_1, ..., w_n)$.** ■

DEFINITION: The orbifold $\mathbb{C}P^{n-1}(w_1, ..., w_n)$ is called **the weighted projective space**.

CLAIM: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, ..., w_n \in \mathbb{Z}^{>0}$, and $Z \subset \mathbb{C}^n \backslash 0$ be a ρ-invariant submanifold. **Then the orbit space $Z/\mathbb{C}^*$ is a projective orbifold in the corresponding weighted projective space $\mathbb{C}P^{n-1}(w_1, ..., w_n)$.**
■

COROLLARY: Let ρ be $\mathbb{C}^*$ acting on $\mathbb{C}^n$ with weights $w_1, ..., w_n \in \mathbb{Z}^{>0}$, and $Z \subset \mathbb{C}^n \backslash 0$ a ρ-invariant submanifold. **Then Z is an open algebraic cone.** ■

Stein varieties equipped with a contraction are algebraic cones

COROLLARY: Let $\tilde M_c$ be a normal Stein variety equipped with a holomorphic contraction γ, with the only singularity at the origin, denoted c. **Then $\tilde M_c$ admits a structure of an algebraic cone.**

Proof. Step 1: Let $\tilde M := \tilde M_c \backslash c$. Then $\frac{\tilde M}{\langle \gamma \rangle}$ admits a holomorphic embedding to a linear Hopf manifold $\frac{{\mathbb C}^n\backslash 0}{\langle A \rangle}$. **This implies that the ideal of $\tilde M_c$ in ${\mathcal O}_{{\mathbb C}^n}$ is generated by A^*-finite functions, that is, polynomials,** hence $\tilde M_c \subset {\mathbb C}^n$ is an affine subvariety.

Step 2: Let ${\mathcal G}_A$ be a connected component of the algebraic closure of $\langle A \rangle$. A connected abelian algebraic group ${\mathcal G}_A \subset GL(n,{\mathbb C})$ is isomorphic to $({\mathbb C}^*)^k \times {\mathbb C}^l$, where each ${\mathbb C}^*$ acts diagonally, and ${\mathbb C}$ are unipotent subgroups. Therefore, $\tilde M_c \subset {\mathbb C}^n$ is ${\mathbb C}^*$-invariant, for some ${\mathbb C}^*$ acting by contractions. **This produces a ${\mathbb C}^*$-fibration $\tilde M {\to} \frac{\tilde M}{{\mathbb C}^*}$ with the quotient $\frac{{\mathbb C}^n\backslash 0}{{\mathbb C}^*}$ identified with the weighted projective space.** ■

Algebraic cones: uniqueness of the complex structure

LEMMA: Let $M \subset H$ be a submanifold in a Hopf manifold, and $\tilde{M}_c \subset \mathbb{C}^n$ the Stein completion of its $\mathbb{Z}$-cover. Denote by $\gamma \in \operatorname{Aut}(\tilde{M}_c)$ the holomorphic contraction generating the $\mathbb{Z}$-action on $\tilde{M}_c$. Choose a compact set $K \subset \tilde{M}_c$ containing an open neighbourhood of the apex. Denote by $\mathcal{B} \subset H^0(\mathcal{O}_{\tilde{M}_c})$ the following ring of functions on $\tilde{M}_c$:

$$\mathcal{B} := \{f \in H^0(\mathcal{O}_{\tilde{M}_c}) \mid \exists\, C > 0 \text{ such that } \forall i \ \ \sup_K |(\gamma^*)^{-i}|f| < C^i\}. \quad (*)$$

Then $\mathcal{B}$ coincides with the space of γ-finite functions.

REMARK: We call a function satisfying (*) **the function of polynomial growth**. This terminology is justified because for γ a linear contraction of $\mathbb{C}^n$, (*) is equivalent to having polynomial growth.

Proof. Step 1: Let f be a γ-finite function, and W the space generated by $\{f, (\gamma^*)f, (\gamma^*)^2 f, ...\}$. Let $\|\cdot\|_K$ be the norm on W defined by $\|f\| := \sup_K |f|$, and let $C := \sup_{\|f\|=1} \|(\gamma^*)^{-1} f\|$ be the operator norm of the map $(\gamma^*)^{-1} \in \operatorname{End}(W)$ in this norm. Then $\sup_K |(\gamma^*)^{-i} f| \leqslant C^i \sup_K |f|$, hence f has polynomial growth.

Step 2: Suppose that f has polynomial growth, and let W be the space of all functions generated by $\{f, (\gamma^*)f, (\gamma^*)^2 f, ...\}$. Then all elements of W have the same growth as f, with the same bound C, hence the closure $\overline{W}$ of W in the norm $\|f\| := \sup_K |f|$ consists of functions with polynomial growth.

The condition (*) holds for all $f \in W$ if and only if $(\gamma^*)^{-1}$ has finite norm on W. Therefore, $\gamma^*\Big|_{\overline{W}}$ is invertible. It remains only to show that the norm of $(\gamma^*)^{-1}$ is infinite on $\overline{W}$ if $\overline{W}$ is infinite-dimensional.

Step 3: The operator γ^* on $\overline{W}$ is compact; by the Riesz-Schauder theorem, it has the Jordan cell decomposition with eigenvalues converging to 0, unless $\overline{W}$ is finite-dimensional. The norm of a linear operator A with eigenvalues α_i satisfies $\|A\| \geqslant \sup |\alpha_i|$. Therefore, a compact operator cannot be invertible on an infinitely-dimensional Banach space: the inverse operator would have infinite norm. ■

COROLLARY: The algebraic structure on $\tilde{M}_c$ is uniquely determined by γ. ■